Chapter 1

Propositional logic

1.1 Prologue: A logic problem on labels and jars

Mathematics is all about solving problems involving objects like sets, functions, numbers, derivatives, integrals, and so on. The goal of this chapter is to train and enhance your problem solving skills in general, by explaining you some tools from mathematical logic. To identify and motivate these tools, we consider as an example the following problem:

Example 1.1.1

Problem: Given are three jars. You cannot see what is inside the bottles, but they are labelled with “Apples”, “Both” and “Pears”. The label “Both” simply means that the jar contains both apples and pears. However, the problem is that someone switched the labels in such a way that no label is on the right jar anymore. In other words: We know that for any jar, it holds that its label is “Apples” or “Both” or “Pears”. Also we know that currently all labels are wrong, which implies that the left jar has true label “Both” or “Pears”, the middle jar has true label “Apples” or “Pears”, while the right jar has true label “Apples” or “Both”.

To figure out where the labels really should be placed, you can draw fruit from each jar. How many times would you need to draw from the jars in order to figure out where the labels were originally?

We will solve this puzzle later, but feel free to think about it already now!

1.2 Getting started with propositional logic

Now the point of the puzzle with the jars and labels is, that thinking about it identifies several key ingredients that are useful in general, when thinking about a mathematical problem. One uses words like “and”, “or”, “not”, “if ... then” when attacking problems of this sort. Let

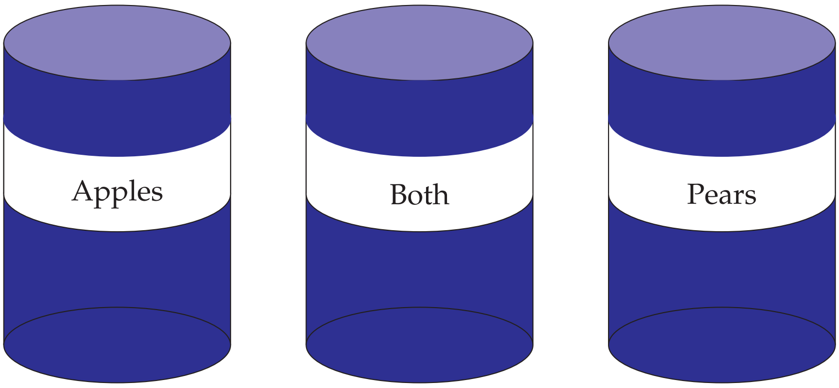

Figure 1.1: A problem with labels and jars

us therefore introduce some notation from what is known as *propositional logic*. First of all, it deals with combinations of short statements that can be either true or false. An example of such a short statement is: the label of jar number one is "Apples". We will call such a statement a logical *proposition*. Here are three more examples of such propositions: $x = 10$, $1 < y$, $a \neq p$.

We will typically use variables like P, Q and so on, to denote such propositions. Saying that a logical proposition P can be true of false, is more formally stated as: P can take the value T (T for true), or the value F (F for false). It is also common to use the number 1 instead of T and 0 instead of F, but in this text we will stick to T and F.

Sometimes a proposition can be broken into smaller, simpler ones. For example, the proposition

$$x = 10 \quad \text{and} \quad 1 < y,$$

consists of the two simpler propositions $x = 10$, $1 < y$ combined with the word 'and'. In propositional logic, one writes

$$x = 10 \quad \wedge \quad 1 < y.$$

To improve readability, one can place paranteses around parts of the expression and for example write:

$$(x = 10) \wedge (1 < y).$$

To be very precise on what $\wedge$ means, let us describe exactly when an expression of the form $P \wedge Q$ is true. We will do this in the following definition

Definition 1.2.1

Let P and Q be two logical propositions. Then $P \wedge Q$, pronounced as "P and Q", is true precisely if P is true and Q is true. In table form:

P	Q	$P \wedge Q$
T	T	T
F	T	F
T	F	F
F	F	F

The table in this definition is called a *truth table* for the logical proposition $P \wedge Q$. Let us explain in more detail how such a truth table works. The two variables P and Q can both be true or false independently of each other. In other words: P and Q can take the value T and F independently. Therefore there are in total four cases to consider:

1) P and Q both take the value T,
2) P takes the value F and Q takes the value T,
3) P takes the value T and Q takes the value F,
4) P and Q both take the value F.

For each of these four possibilities the truth table of $P \wedge Q$ specifies which value $P \wedge Q$ takes. For example if P takes the value T and Q takes the value F, we read off from the third row of the truth table that $P \wedge Q$ takes the value F This is why the truth table of $P \wedge Q$ has four rows. Each row specifies which value $P \wedge Q$ takes if P and Q take specific values.

More complicated logical propositions also have a truth table. Here is one example:

Example 1.2.1

Let P, Q, R be three logical propositions. Now consider the logical proposition $P \wedge (Q \wedge R)$. We have put parentheses around $Q \wedge R$ to clarify that we consider P combined with $Q \wedge R$ using $\wedge$. The logical proposition $(P \wedge Q) \wedge R$ may look similar, but is strictly speaking not the same as $P \wedge (Q \wedge R)$!

To determine when $P \wedge (Q \wedge R)$ is true and when it is false, we use Definition 1.2.1 and compute its truth table. Since we have three variables now, the truth table will contain eight rows: one row for each possible value taken by P, Q, and R. Therefore the table starts like this:

P	Q	R
T	T	T
F	T	T
T	F	T
F	F	T
T	T	F
F	T	F
T	F	F
F	F	F

Since $P \wedge (Q \wedge R)$ consists of P and $Q \wedge R$, it is convenient to first add a column concerning $Q \wedge R$. To fill out the values $Q \wedge R$ takes in each of the eight rows, we use Definition 1.2.1. Indeed, even though in Definition 1.2.1 the logical propositions were called P and Q, we can also apply it for the logical proposition Q and R. For example, in the first two rows, both Q and R take the value T, which according to Definition 1.2.1 means that then also $Q \wedge R$ takes the value T. In the third and fourth row Q takes the value F and R the value T. Hence in these rows, $Q \wedge R$ takes the value F. Continuing like this, we then obtain:

P	Q	R	$Q \wedge R$
T	T	T	T
F	T	T	T
T	F	T	F
F	F	T	F
T	T	F	F
F	T	F	F
T	F	F	F
F	F	F	F

Next we add a column for $P \wedge (Q \wedge R)$ and determine the truth values it takes for each of the eight rows. Suppose for example that P, Q, R take the values F, T, T. This correspond to the values specified in the second row of the truth table. In that case, we see from the column that we have just computed, that $Q \wedge R$ takes the value T. But then applying Definition 1.2.1 for the logical propositions P and $Q \wedge R$, we see that $P \wedge (Q \wedge R)$ takes the value F. Continuing like this, we can compute the final column for $P \wedge (Q \wedge R)$ and complete the truth table:

P	Q	R	$Q \wedge R$	$P \wedge (Q \wedge R)$
T	T	T	T	T
F	T	T	T	F
T	F	T	F	F
F	F	T	F	F
T	T	F	F	F
F	T	F	F	F
T	F	F	F	F
F	F	F	F	F

We can think of $\wedge$ as a logical operator: given two logical propositions P and Q, no matter how complicated P and Q already are, it produces a new logical proposition $P \wedge Q$. In this light $\wedge$ is sometimes called the *conjunction* and $P \wedge Q$ called the conjunction of P and Q.

Let us now introduce more logical operators. In Example 1.1.1, we knew that all labels were wrong initially. Hence, the first jar on the left does not have label “Apples”. This means that it has label “Both” or “Pears”. This is formalized in the next definition:

Definition 1.2.2

Let P and Q be two propositions. Then $P \vee Q$, pronounced as “P or Q", is defined by the following truth table:

P	Q	$P \vee Q$
T	T	T
F	T	T
T	F	T
F	F	F

The operator $\vee$ is called *disjunction* and $P \vee Q$ the disjunction of P and Q. A further logical operator is the negation of a logical proposition. We have already used this as well in Example 1.1.1. There we said that the labels were wrong. In particular we know that the true label of the middle jar was not "Both". Also a proposition like $x \neq 0$ is simply the negation of the proposition $x = 0$. We now formally define the negation operator.

Definition 1.2.3

Let P be a proposition. Then $\neg P$, pronounced as "not P", is defined by the following truth table:

P	$\neg P$
T	F
F	T

As operator, $\neg$ is called the *negation*, and $\neg P$ is therefore also called the negation of P. We now already have enough ingredients to create various logical propositions. Let us consider an example.

Example 1.2.2

Consider the logical proposition $P \vee (Q \wedge \neg P)$. We determine its truth table. Having only two variables P and Q, this truth table will contain four rows. Further $P \vee (Q \wedge \neg P)$ contains the simpler logical proposition $Q \wedge \neg P$, which in turn contains the logical proposition $\neg P$. Therefore, when computing the truth table of $P \vee (Q \wedge \neg P)$, it makes sense to add a column for $\neg P$ and one for $Q \wedge \neg P$ and in this way gradually work our way towards computing the truth values of the whole logical proposition $P \vee (Q \wedge \neg P)$. Then the result is the following:

P	Q	$\neg P$	$Q \wedge \neg P$	$P \vee (Q \wedge \neg P)$
T	T	F	F	T
F	T	T	T	T
T	F	F	F	T
F	F	T	F	F

Let us compare the truth table we just computed and the truth table of $P \vee Q$ from Definition 1.2.2. The comparison shows that for given truth values of P and Q, the truth values of $P \vee (Q \wedge \neg P)$ and $P \wedge Q$ are always the same! In other words: if we take the three columns of the truth table we just computed corresponding to P, Q and $P \vee (Q \wedge \neg P)$, then we get precisely the same table as the truth tabel from Definition 1.2.2. Apparently, two different looking logical propositions, can have the same truth tables.

1.3 Logical consequence and equivalence

The logical operators we introduced so far, $\neg$, $\wedge$ and $\vee$, allow us to write down a variety of logical propositions in a precise way. However, the whole point of logic is to make arguments and reasoning more precise. We would like to be able to say something like, if P is true, then we may conclude that Q also is true. For example, if $x > 0$, then also $x > -1$. To formalize this, we use the logical symbol $\Rightarrow$, called an *implication*, and write $P \Rightarrow Q$. We define it by giving its truth table.

Definition 1.3.1

The logical proposition $P \Rightarrow Q$ is defined by the following truth table:

P	Q	$P \Rightarrow Q$
T	T	T
F	T	T
T	F	F
F	F	T

In common language one often pronounces $P \Rightarrow Q$ as "P implies Q" or "if P then Q". It is sometimes convenient to write the logical proposition $P \Rightarrow Q$ as $Q \Leftarrow P$.

There are two special types of logical propositions that are simply denoted by **T** and **F**. The logical proposition **T** simply stands for a statement that is always true, like for example the statement $5 = 5$. Such a logical proposition is called a *tautology*. By contrast, the logical proposition **F**, stands for a statement that is always false, like for example $5 \neq 5$. This is called a *contradiction*. Going back to implications, saying that $P \Rightarrow Q$ is always true for certain logical propositions P and Q, really means that we claim that $P \Rightarrow Q$ is a tautology. If $P \Rightarrow Q$ is a tautology, then the truth table of the implication from Definition 1.3.1, shows that P is true implies that Q is true as well.

If $P \Rightarrow Q$ is a tautology, then one says that Q is a *logical consequence* of P, or alternatively that Q is implied by P. This explains why the symbol $\Rightarrow$ is called an implication. Let us consider an example of a logical consquence.

Example 1.3.1

Let P and Q be logical propositions. Then we claim that the logical proposition $P \vee Q$ is a logical consequence of P. To show this, we need to verify that the logical proposition $P \Rightarrow (P \vee Q)$ is always true. In other words, we need to show that $P \Rightarrow (P \vee Q)$ is a tautology. In order to compute the truth table of $P \Rightarrow (P \vee Q)$, we proceed in the usual way. First we write down all combinations of truth values of P and Q:

P	Q
T	T
F	T
T	F
F	F

Next, we add a column for $P \vee Q$ for convenience, since it occurs in the more complicated proposition $P \Rightarrow (P \vee Q)$ we are looking at. Using Definition 1.2.2, we then find the following:

P	Q	$P \vee Q$
T	T	T
F	T	T
T	F	T
F	F	F

Now we add a column for $P \Rightarrow (P \vee Q)$ and use Definition 1.3.1 to compute its truth values from the ones of P and $P \vee Q$. The result is the truth table we wanted to compute:

P	Q	$P \vee Q$	$P \Rightarrow (P \vee Q)$
T	T	T	T
F	T	T	T
T	F	T	T
F	F	F	T

Since the column below the expression $P \Rightarrow (P \vee Q)$ only contains T's, we can indeed conclude that $P \Rightarrow (P \vee Q)$ is a tautology. In particular, we can now be sure that P implies $P \vee Q$. Another way of saying this is that $P \vee Q$ is a logical consequence of P.

Stronger than an implication is what is known as a *bi-implication*, denoted by $\Leftrightarrow$ and defined as:

Definition 1.3.2

The logical proposition $P \Leftrightarrow Q$, pronounced as "P if and only if Q", is defined by the following truth table:

P	Q	$P \Leftrightarrow Q$
T	T	T
F	T	F
T	F	F
F	F	T

The phrase "P if and only if Q" for the logical proposition $P \Leftrightarrow Q$ can be broken up in two parts "P if Q" and "P only if Q". The first part, "P if Q" is just a way of saying that $P \Leftarrow Q$, while "P only if Q" boils down to the statement $P \Rightarrow Q$. This explains that name bi-implication for the symbol $\Leftrightarrow$: it in fact combines two implications in one symbol. We will see later in Theorem 1.3.4, Equation (1.22) in a more formal way that a bi-implication can indeed in this way be expressed as two implications.

Example 1.3.2

In Example 1.2.2 we noted that the truth tables of $P \vee Q$ is identical to that of $P \vee (Q \wedge \neg P)$. What does this mean for the truth table of the logical proposition $(P \vee Q) \Leftrightarrow (P \vee (Q \wedge \neg P))$? Using Definition 1.2.2 and Example 1.2.2, we see that the following table is correct:

P	Q	$P \vee Q$	$P \vee (Q \wedge \neg P)$
T	T	T	T
F	T	T	T
T	F	T	T
F	F	F	F

Now let us add a column to this table for the logical proposition $(P \vee Q) \Leftrightarrow (P \vee (Q \wedge \neg P))$ and use Definition 1.3.2. We obtain:

P	Q	$P \vee Q$	$P \vee (Q \wedge \neg P)$	$(P \vee Q) \Leftrightarrow (P \vee (Q \wedge \neg P))$
T	T	T	T	T
F	T	T	T	T
T	F	T	T	T
F	F	F	F	T

Since the rightmost column only contains T, we can conclude that $(P \vee Q) \Leftrightarrow (P \vee (Q \wedge \neg P))$ is a tautology.

The point now is that if $R \Leftrightarrow S$ is a tautology for some, possibly complicated, logical propositions R and S, then the truth tables of R and S are the same. In other words: if R is true, then S is true as well, but also the converse holds: if S is true, then R is true as well. Therefore, if $R \Leftrightarrow S$ is a tautology, one says that the logical propositions R and S are *logically equivalent*. From Example 1.3.2, we can conclude that the logical propositions $P \vee Q$ and $P \vee (Q \wedge \neg P)$ are logically equivalent. The point of this example is that it shows that sometimes one can rewrite a logical statement in a simpler form. There are several convenient tautologies that can be used to rewrite logical propositions in a simpler form. We start by giving some involving conjunction, disjunction and negation.

Theorem 1.3.1

Let P, Q and R be logical propositions. Then all the following expressions are tautologies.

$$P \wedge P \Leftrightarrow P \tag{1.1}$$
$$P \vee P \Leftrightarrow P \tag{1.2}$$
$$P \wedge Q \Leftrightarrow Q \wedge P \tag{1.3}$$
$$P \vee Q \Leftrightarrow Q \vee P \tag{1.4}$$
$$P \wedge (Q \wedge R) \Leftrightarrow (P \wedge Q) \wedge R \tag{1.5}$$
$$P \vee (Q \vee R) \Leftrightarrow (P \vee Q) \vee R \tag{1.6}$$
$$P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R) \tag{1.7}$$
$$P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R) \tag{1.8}$$

Proof. To prove that one of the mentioned logical propositions is a tautology, we compute a truth table for it. Doing this for all of them would fill quite a few pages, but let us consider one of them, namely Equation (1.5). We need to show that $P \wedge (Q \wedge R) \Leftrightarrow (P \wedge Q) \wedge R$ is a tautology. In Example 1.2.1, we already computed the truth table of $P \wedge (Q \wedge R)$, so we do not have to redo that here. What we will need to do is to compute the truth table of $(P \wedge Q) \wedge R$, in a way similar to what we did for $P \wedge (Q \wedge R)$ in Example 1.2.1, and then in the last step compute the truth table of $P \wedge (Q \wedge R) \Leftrightarrow (P \wedge Q) \wedge R$ using Definition 1.3.2. The result is the following:

P	Q	R	$P \wedge (Q \wedge R)$	$P \wedge Q$	$(P \wedge Q) \wedge R$	$P \wedge (Q \wedge R) \Leftrightarrow (P \wedge Q) \wedge R$
T	T	T	T	T	T	T
F	T	T	F	F	F	T
T	F	T	F	F	F	T
F	F	T	F	F	F	T
T	T	F	F	T	F	T
F	T	F	F	F	F	T
T	F	F	F	F	F	T
F	F	F	F	F	F	T

We see that the logical proposition $P \wedge (Q \wedge R) \Leftrightarrow (P \wedge Q) \wedge R$ only takes the truth value T, no matter what values P, Q and R take. Hence we can conclude that $P \wedge (Q \wedge R) \Leftrightarrow (P \wedge Q) \wedge R$ is a tautology.

All the other items in the theorem can be shown similarly, but we will not do so here. Readers are encouraged to prove at least one other item themselves. □

In words, Equation (1.6) states that when taking the disjunction of three logical propositions, it does not matter how you place the parentheses. Therefore, it is common to write $P \vee Q \vee R$ and leave the parentheses out completely. Similarly Equation (1.5) says that for the conjunction of three logical propositions, you can place the parentheses as you want. Therefore, on can write $P \wedge Q \wedge R$ without any ambiguity. This situation changes if both conjunction and disjunction occur in the same expression. Then parentheses do matter. We consider an example.

Example 1.3.3

Consider the logical propositions $(P \wedge Q) \vee R$ and $P \wedge (Q \vee R)$. We claim that these are not logically equivalent. To show this, we could compute their truth tables, but in fact to show that two logical propositions are not logically equivalent, all we need to do is to find values for P, Q and R such that $(P \wedge Q) \vee R$ and $P \wedge (Q \vee R)$ are not both true. Let us for example find out when $(P \wedge Q) \vee R$ is false. This happens precisely if $P \wedge Q$ is false and R is false. Hence $(P \wedge Q) \vee R$ is false precisely if P and Q are not both true and R is false. However, $P \wedge (Q \vee R)$ will be false whenever P is false. Hence if (P, Q, R) take the values (F, T, T), then $(P \wedge Q) \vee R$ is true, but $P \wedge (Q \vee R)$ is false. This means that in the truth table of the two expressions, there is a row looking as follows:

P	Q	R	...	$(P \wedge Q) \vee R$	$P \wedge (Q \vee R)$
$\vdots$					
F	T	T	...	T	F
$\vdots$					

This is in fact enough to conclude that the logical propositions $(P \wedge Q) \vee R$ and $P \wedge (Q \vee R)$ are not logically equivalent. Indeed, if they would be, the logical proposition $(P \wedge Q) \vee R \Leftrightarrow P \wedge (Q \vee R)$ would be a tautology and hence only take the value T, but based on the previous, we see that its truth table actually contains the following row:

P	Q	R	...	$(P \wedge Q) \vee R$	$P \wedge (Q \vee R)$	$(P \wedge Q) \vee R \Leftrightarrow P \wedge (Q \vee R)$
$\vdots$						
F	T	T	...	T	F	F
$\vdots$						

This shows that $(P \wedge Q) \vee R \Leftrightarrow P \wedge (Q \vee R)$ is not a tautology and therefore that the logical propositions $(P \wedge Q) \vee R$ and $P \wedge (Q \vee R)$ indeed are not logically equivalent.

There are a few more tautologies that are useful when dealing with logical propositions. Apart from the conjunction $\wedge$ and disjunction $\vee$, these also involve the negation $\neg$. We leave the proofs to the reader.

Theorem 1.3.2

Let P, Q and R be logical propositions. Then all the following expressions are tautologies.

$$P \vee \neg P \Leftrightarrow \mathbf{T} \tag{1.9}$$
$$P \wedge \neg P \Leftrightarrow \mathbf{F} \tag{1.10}$$
$$P \Leftrightarrow \neg(\neg P) \tag{1.11}$$
$$\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q \tag{1.12}$$
$$\neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q \tag{1.13}$$
$$\neg\mathbf{T} \Leftrightarrow \mathbf{F} \tag{1.14}$$
$$\neg\mathbf{F} \Leftrightarrow \mathbf{T} \tag{1.15}$$

Identities (1.12) and (1.13) are called the *De Morgan's laws*. Finally, there are a few tautologies describing how $\wedge$ and $\vee$ interact with tautologies and contradictions. Again, we leave the proofs of these to the reader.

Theorem 1.3.3

Let P, Q and R be logical propositions. Then all the following expressions are tautologies.

$$P \vee \mathbf{F} \Leftrightarrow P \tag{1.16}$$
$$P \wedge \mathbf{T} \Leftrightarrow P \tag{1.17}$$
$$P \wedge \mathbf{F} \Leftrightarrow \mathbf{F} \tag{1.18}$$
$$P \vee \mathbf{T} \Leftrightarrow \mathbf{T} \tag{1.19}$$

Using the list of tautologies in Theorems 1.3.1, 1.3.2 and 1.3.3 one can rewrite logical proposition in a logically equivalent form. Let us consider an example.

Example 1.3.4

As in Examples 1.2.2 and 1.3.2, consider the logical proposition $P \vee (Q \wedge \neg P)$. We have already seen that it is logically equivalent to $P \vee Q$, but let us now show this using Theorem 1.3.1 and not by computing truth tables. First of all, using (1.8), we see that

$$P \vee (Q \wedge \neg P) \Leftrightarrow (P \vee Q) \wedge (P \vee \neg P).$$

Using (1.9), we conclude that

$$P \vee (Q \wedge \neg P) \Leftrightarrow (P \vee Q) \wedge \mathbf{T},$$

which by (1.17) can be simplified to

$$P \vee (Q \wedge \neg P) \Leftrightarrow P \vee Q.$$

In other words, using Theorem 1.3.1, one can prove logical equivalences without having to compute truth tables. Of course when proving this theorem, one needs to compute several truth tables, but this only needs to be done once. Generally speaking in mathematics, the point of a theorem is that it contains one or several useful results with a proof. Once the proof is given, one can use the result in the theorem whenever needed without having to prove the theorem again.

The tautologies in Theorem 1.3.1 only involve negation, conjunction and disjunction. Here are three very useful ones that involve implication and bi-implication as well.

Theorem 1.3.4

Let P and Q be logical propositions. Then all the following expressions are tautologies.

$$(P \Rightarrow Q) \Leftrightarrow (\neg P \vee Q) \tag{1.20}$$
$$(P \Rightarrow Q) \Leftrightarrow (\neg Q \Rightarrow \neg P) \tag{1.21}$$
$$(P \Leftrightarrow Q) \Leftrightarrow (P \Rightarrow Q) \wedge (Q \Rightarrow P) \tag{1.22}$$
$$P \Leftrightarrow (\neg P \Rightarrow \mathbf{F}) \tag{1.23}$$

Proof. As in Theorem 1.3.1, these items can be shown by computing truth tables for each of them. We will do this for the second item and leave the others to the reader:

P	Q	$P \Rightarrow Q$	$\neg P$	$\neg Q$	$\neg Q \Rightarrow \neg P$	$(P \Rightarrow Q) \Leftrightarrow (\neg Q \Rightarrow \neg P)$
T	T	T	F	F	T	T
F	T	T	T	F	T	T
T	F	F	F	T	F	T
F	F	T	T	T	T	T

Since the right column only contains T, we conclude that $(P \Rightarrow Q) \Leftrightarrow (\neg Q \Rightarrow \neg P)$ indeed is a tautology. $\square$

Equation (1.20) means that in principle, an implication can be expressed using negation and disjunction. Equation (1.21) is called *contraposition*. It means that if one wants to prove that Q is a logical consequence of P, it is also fine to show that $\neg P$ is a logical consequence of $\neg Q$. Let us consider a small example of contraposition.

Example 1.3.5

Consider the statement that for any real numbers x and y it holds that

$$(x \cdot y = 0) \Rightarrow ((x = 0) \vee (y = 0)).$$

This is a true statement, but in this example we do not want to prove it, but simply to figure out what the contraposition of this statement is.

First of all, the given statement is a logical proposition of the form $P \Rightarrow Q$, where P is the equation $x \cdot y = 0$ and Q the proposition $(x = 0) \vee (y = 0)$. In words, the implication $P \Rightarrow Q$ can be phrased as: if for some real numbers x and y, the equation $x \cdot y = 0$ holds, then $x = 0$ or $y = 0$.

What is the contraposition of this? According to Equation (1.21) it is $\neg Q \Rightarrow \neg P$. When used directly, we therefore find that the contraposition we are looking for, is

$$\neg((x = 0) \vee (y = 0)) \Rightarrow \neg(x \cdot y = 0).$$

However, we can simplify this a bit. First of all, one can rewrite $\neg(x \cdot y = 0)$ as $x \cdot y \neq 0$. Moreover, using Equation (1.12), one of the DeMorgan laws, we can rewrite $\neg((x = 0) \vee (y = 0))$ as $\neg(x = 0) \wedge \neg(y = 0)$, which in turn can be written as $(x \neq 0) \wedge (y \neq 0)$. Therefore the contraposition of

$$(x \cdot y = 0) \Rightarrow (x = 0) \vee (y = 0)$$

can be given as

$$((x \neq 0) \wedge (y \neq 0)) \Rightarrow (x \cdot y \neq 0).$$

More in words: the contraposition of the statement "if $x \cdot y = 0$, then $x = 0$ or $y = 0$" simply is "if $x \neq 0$ and $y \neq 0$, then $x \cdot y \neq 0$".

This last statement is a true statement, since it is logically equivalent to the true statement that we started with in this example.

Equation (1.22) states that two logical propositions are logically equivalent precisely if they are logical consequences of each other. Quite often it is easier to show that $P \Rightarrow Q$ and $Q \Rightarrow P$ are true separately, then to show directly that $P \Leftrightarrow Q$ is true. Also Equation 1.23 is sometimes used to prove logical statements: instead of showing that P is true, one assumes that P is false and then tries to obtain a contradiction. If one does obtain a contradiction, one can conclude that $\neg P \Rightarrow \mathbf{F}$ is true. But then by Equation (1.23), P is also true. This method is called a proof by contradiction.

In later chapters, we will regularly use Equations (1.21), (1.22) and (1.23), when investigating various mathematical statements. In the next section, we will also show uses of logic in mathematics.

1.4 Use of logic in mathematics

Logic can help to solve mathematical problems and to clarify the mathematical reasoning. In this section, we give a number of examples of this.

Example 1.4.1

Question: Determine all real numbers x such that $-x \leq 0 \leq x - 1$.

Answer: $-x \leq 0 \leq x - 1$ is really shorthand for the logical proposition

$$-x \leq 0 \quad \wedge \quad 0 \leq x - 1.$$

The first inequality is logically equivalent to the inequality $x \geq 0$, while the second one is equivalent to $x \geq 1$. Hence a real number x is a solution if and only if

$$x \geq 0 \quad \wedge \quad x \geq 1.$$

The answer is therefore all real numbers x such that $x \geq 1$.

Example 1.4.2

Question: determine all real numbers x such that $2|x| = 2x + 1$. Here $|x|$ denotes the absolute value of x.

Answer: if $x < 0$, then $|x| = -x$, while if $x \geq 0$, then $|x| = x$. Hence it is convenient to consider the cases $x < 0$ and $x \geq 0$ separately. More formally, we have the following sequence of logically equivalent statements:

$$
\begin{array}{lcl}
2|x| = 2x + 1 & & \\
\Leftrightarrow & & \\
2|x| = 2x + 1 & \wedge & (x < 0 \quad \vee \quad x \geq 0) \\
\Leftrightarrow & & \\
(2|x| = 2x + 1 \quad \wedge \quad x < 0) & \vee & (2|x| = 2x + 1 \quad \wedge \quad x \geq 0) \\
\Leftrightarrow & & \\
(-2x = 2x + 1 \quad \wedge \quad x < 0) & \vee & (2x = 2x + 1 \quad \wedge \quad x \geq 0) \\
\Leftrightarrow & & \\
(-4x = 1 \quad \wedge \quad x < 0) & \vee & (0 = 1 \quad \wedge \quad x \geq 0) \\
\Leftrightarrow & & \\
(x = -1/4 \quad \wedge \quad x < 0) & \vee & (\mathbf{F} \quad \wedge \quad x \geq 0) \\
\Leftrightarrow & & \\
x = -1/4 \quad \vee \quad \mathbf{F} & & \\
\Leftrightarrow & & \\
x = -1/4 & &
\end{array}
$$

Hence the only solution to the equation $2|x| = 2x + 1$ is $x = -1/4$.

Example 1.4.3

Question: Determine all nonnegative real numbers such that $\sqrt{x} = -x$.

Observation: It is tempting to take the square on both sides, one then obtains $x = x^2$, and then to conclude that $x = 0$ and $x = 1$ are the solutions to the equation $\sqrt{x} = -x$. However, $x = 0$ is indeed a solution, but $x = 1$ is not, since $\sqrt{1} \neq -1$. What went wrong?

Answer: The reasoning actually shows that if x satisfies the equation $\sqrt{x} = -x$, then $x = x^2$, which in turn implies that $x = 0$ or $x = 1$. Hence the following statement is completely correct:

$$(\sqrt{x} = -x) \Rightarrow (x = 0 \vee x = 1).$$

In that sense, nothing went wrong and any solution to the equation $\sqrt{x} = -x$ must indeed be either $x = 0$ or $x = 1$. What may cause confusion is that this does not at all mean that $x = 0$ and $x = 1$ both are solutions to the equation $\sqrt{x} = -x$. This would namely amount to the statement

$$(x = 0 \vee x = 1) \Rightarrow (\sqrt{x} = -x),$$

which is different from what we have shown and actually is not true. To solve the question, all we need to do it to check if the potential solutions $x = 0$ and $x = 1$ really are solutions. We then obtain that $x = 0$ is the only solution.

1.5 Epilogue: the logic problem on labels and jars

Let us return to the problem of jars and labels from the first section.

Example 1.5.1

Let us denote by $P_1(A)$ the statement that the left jar has true label "Apples". Similarly, let us write $P_1(B)$, respectively $P_1(P)$, for the statement that the left jar has true label "Both", respectively "Pears". We then know that $P_1(B) \vee P_1(P)$ is always true, since the left jar cannot have label "Apples". Similarly for the middle jar, we can introduce $P_2(A)$, $P_2(B)$, and $P_2(P)$ for the statements that the middle jar has true label "Apples", "Both", "Pears" and conclude that $P_2(A) \vee P_2(P)$ is a true statement. Similarly for the right jar, we obtain that $P_3(A) \vee P_3(B)$ is a true statement. In conclusion,

$$(P_1(B) \vee P_1(P)) \wedge (P_2(A) \vee P_2(P)) \wedge (P_3(A) \vee P_3(B)) \tag{1.24}$$

is always true. Using Equation (1.7) repeatedly, we can rewrite this to the logically equivalent statement

$$\begin{array}{lll}
(P_1(B) \wedge P_2(A) \wedge P_3(A)) & \vee & (P_1(B) \wedge P_2(A) \wedge P_3(B)) \quad \vee \\
(P_1(B) \wedge P_2(P) \wedge P_3(A)) & \vee & (P_1(B) \wedge P_2(P) \wedge P_3(B)) \quad \vee \\
(P_1(P) \wedge P_2(A) \wedge P_3(A)) & \vee & (P_1(P) \wedge P_2(A) \wedge P_3(B)) \quad \vee \\
(P_1(P) \wedge P_2(P) \wedge P_3(A)) & \vee & (P_1(P) \wedge P_2(P) \wedge P_3(B)).
\end{array}$$

This statement is still valid, since it is logically equivalent to the statement from Equation 1.24. Since we know that in the correct labelling each label has to be used exactly once, a statement like $P_1(B) \wedge P_2(A) \wedge P_3(A)$ where the same label occurs twice, cannot be correct, that is to say that it is a contradiction. Using that disjunction absorbs contradictions, see Equation (1.16), we therefore conclude that

$$(P_1(B) \wedge P_2(P) \wedge P_3(A)) \quad \vee \quad (P_1(P) \wedge P_2(A) \wedge P_3(B)) \tag{1.25}$$

is necessarily always true.

What this shows is that there are only two possible correct ways to label the jars. This is already very helpful, since we did not even draw any fruit yet! Now let us investigate what the effect of drawing from a jar is. If we draw from the left jar, we do not learn much about the label of that jar. Indeed, since the true label is "Both" or "Pears", if we draw an apple from it, we know the true label cannot be "Pears", but if we draw a pear from it, the true label could still be "Both" or "Pears". Similarly drawing from the right jar, may not determine its true label. The situation is different for the middle jar. Since the true label of the middle jar is "Apples" or "Pears", if we draw an apple from it, its true label cannot be "Pears". Apparently, it must be "Apples" in that case. Similarly, if we draw a pear from the middle jar, its true label is "Pears". We arrive at the following solution for the problem:

Solution:

Step 1: Draw from the middle jar. Since we know all labels are wrong, the middle jar, that has label "Both", contains either only apples, or only pears. If we draw an apple from the middle jar, then we can conclude the correct label should have been "Apples," while if we draw a pear from the middle jar, then we can conclude that that correct label should have been "Pears".

Step 2: We know that the logical proposition in Equation 1.25 is always true. This implies that if we found in Step 1 that the correct label for the middle jar is "Apples", then $P_1(P) \wedge P_2(A) \wedge P_3(B)$ is true, while if the correct label of the middle jar was identified as "Pears" in Step 1, then $P_1(B) \wedge P_2(P) \wedge P_3(A)$ is true.

Conclusion: We only need to draw once! After that we can identify all three labels correctly. Moreover, we have actually found a simple step-by-step procedure to determine the correct labelling. This is an example of what one calls an algorithm. To make it look more like a computer algorithm, we give it as follows:

Algorithm 1 Label Identifier

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1: Draw from the jar labelled "Both" and denote the result by R.
2: if R = apple then
3:     Identify the labels of the jars as "Pears","Apples","Both",
4: else
5:     Identify the labels of the jars as "Both","Pears","Apples".
```

There are many puzzles of this type. Here is another one. Feel free to try to solve it yourself before reading the solution.

Example 1.5.2

A police officer is investigating a burglary and was able to narrow the number of suspects down to three. The officer is absolutely sure that one of these three committed the crime and that the perpetrator worked alone. When questioning the three suspects, the following statements are made by the suspects:

Suspect1:	"Suspect2 did it";
	"I wasn't there";
	"I am innocent"
Suspect2:	"Suspect3 is innocent";
	"everything Suspect 1 said is a lie";
	"I didn't do it"
Suspect3:	"I didn't do it";
	"Suspect1 is lying if he said that he wasn't there";
	"Suspect2 is lying if he said that everything that Suspect1 said is a lie"

Confused, the police officer goes to the boss, the police commissioner. The police commissioner says: "I know these suspects quite well and every single one of them always lies at least once in their statements." Can you help the police officer to figure out which suspect is guilty of the burglary?

Solution Let us introduce some logical proposition to analyze the situation. First of all, P_1 is the statement "Suspect1 did it" and similarly P_2 stands for "Suspect2 did it", P_3 for "Suspect3 did it". With this notation in place, we know that

$$P_1 \vee P_2 \vee P_3$$

is true, since the police officer is absolutely sure that one of the three suspects committed the burglary.

Now let us analyze the statements from the suspects:

Statements from Suspect1:	
"Suspect2 did it";	this is just P_2
"I wasn't there";	we call this R_1
"I am innocent";	this amounts to $\neg P_1$

Now let us consider the insight from the police commissioner: any of the three suspects has lied at least once in their statements. In particular, Suspect1 is lying, which means that $\neg P_2 \vee \neg R_1 \vee \neg(\neg P_1)$ is a true statement. Using Equation (1.11), we conclude that

$$\neg P_2 \vee \neg R_1 \vee P_1$$

is a true statement as well.

Statements from Suspect2:	
"Suspect3 is innocent";	this is $\neg P_3$
"everything Suspect 1 said is a lie";	this amount to $\neg P_2 \wedge \neg R_1 \wedge P_1$
"I didn't do it";	this is $\neg P_2$

Now let us again consider the insight from the police commissioner. For Suspect 2 we obtain that $P_3 \vee \neg(\neg P_2 \wedge \neg R_1 \wedge P_1) \vee P_2$ is a true statement. One can simplify this expression using Theorem 1.3.1. First of all, using Equation (1.13), the proposition $\neg(\neg P_2 \wedge \neg R_1 \wedge P_1)$ is logically equivalent to $\neg(\neg P_2) \vee \neg(\neg R_1) \vee \neg P_1)$, which in turn is logically equivalent to $P_2 \vee R_1 \vee \neg P_1$ using Equation (1.11). Substituting this in the original statement, we see that $P_3 \vee (P_2 \vee R_1 \vee \neg P_1) \vee P_2$ is a true statement. Simplifying $P_2 \vee P_2$ to P_2 using Equation (1.2), we obtain that

$$P_3 \vee P_2 \vee R_1 \vee \neg P_1$$

is a true statement as well.

The statements of Suspect3 are a bit involved, so before putting them in a table, let us consider the last two statements. The second statement of Suspect3 is that "Suspect1 is lying if he said that he wasn't there". In other words: "Suspect1 wasn't there" $\Rightarrow$ "Suspect1 is lying". However, the police commissioner already told us that the statement "Suspect1 is lying" always is true. This means that the implication, "Suspect1 wasn't there" $\Rightarrow$ "Suspect1 is lying", is a true statement. Similarly, the third statement from Suspect3, "Suspect2 is lying if he said that everything that Suspect1 said is a lie", is true. Hence the second and third statements from Suspect3 do not give us any information that we did not already know.

Statements from Suspect3:	
"I didn't do it";	this is $\neg P_3$
"Suspect1 is lying if he said that he wasn't there";	
"Suspect2 is lying if he said that everything that Suspect1 said is a lie;	

Now let us for the third time consider the insight from the police commissioner. First of all, the given insight implies that the second are third statements from Suspect3 are true, since we know that Suspect1 and Suspect2 are lying. Since by the same insight, Suspect3 lied, we conclude that $\neg P_3$ must be a lie. In other words, P_3 must be true.

Collecting everything together, we have determined that the following are all true: $P_1 \vee P_2 \vee P_3$, $\neg P_2 \vee \neg R_1 \vee P_1$, $P_3 \vee P_2 \vee R_1 \vee \neg P_1$, P_3. The fact that P_3 is true, immediately implies that the only possibility is that Suspect3 has committed the burglary and that as a consequence Suspect1 and Suspect2 are innocent. However, we still need to check that in this case all the other statement we obtained are indeed true. If not, this would mean that no solution exists and that the police officer or the police commissioner is wrong. First of all, if P_3 takes the value T, then $P_1 \vee P_2 \vee P_3$ and $P_3 \vee P_2 \vee R_1 \vee \neg P_1$ will be true by the definition of the disjunction. This leaves $\neg P_2 \vee \neg R_1 \vee P_1$. Since Suspect2 is innocent, P_2 takes the value F and as a consequence, $\neg P_2$ takes the value T. Hence indeed $\neg P_2 \vee \neg R_1 \vee P_1$ is a true statement. This means that there is nothing contradictory. The police should arrest Suspect3!