

# The Technical University of Denmark

Page 1 of 6 pages

Written exam, December 7, 2025

Course name: Mathematics 1a (polytechnical foundation)

Course no. 01003

Exam duration: 4 hours

Aids: Written materials permitted

Weight: The problem sheet consists of two parts: a multiple-choice (MC) part covering MC-Problems 1-8, and a part covering Problems 9-15 where answers are to be accompanied by reasoning.

The two parts of this exam weigh equally. The individual MC-Problems in the MC part weigh equally; a wrong or missing answer gives 0 points. In the second part, all sub-questions weigh equally.

**Further information:** Only answers should be provided for the MC problems. In the second part, all answers should be well-reasoned, and intermediate calculations should be provided to an appropriate extent.

**MC-Problem 1** We are given the matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 4 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

Which of the numbers below is the determinant of  $\mathbf{A}$ ?

- 1a)  $-5$     1b)  $1$     1c)  $0$     1d)  $6$     1e)  $-7$     1f)  $3$     1g)  $10$

**MC-Problem 2** We are given the homogeneous second-order differential equation

$$3f''(t) - 6f'(t) + 15f(t) = 0.$$

Which of the expressions below is a solution to the differential equation?

- 2a)  $e^{2t} \sin(t)$                       2b)  $e^t \cos(2t)$                       2c)  $\sin(2t) - \cos(2t)$                       2d)  $1$   
2e)  $e^{3t}(\cos(t) + 2\sin(t))$                       2f)  $5e^{2t} \sin(t)$                       2g)  $e^{-t} \cos(-2t)$

**MC-Problem 3** We are given the polynomial

$$p(Z) = Z^3 - 2Z^2 - Z + 7.$$

Which of the numbers below is the remainder after division by  $Z + 1$ ?

- 3a)  $2$     3b)  $-2$     3c)  $0$     3d)  $5$     3e)  $-5$     3f)  $1$     3g)  $-1$

**MC-Problem 4** Which of the following six functions is bijective?

- 4a)  $f_1 : \mathbb{R} \rightarrow \mathbb{R}, \quad x \mapsto \cos(x),$   
4b)  $f_2 : \mathbb{R} \rightarrow [-1, 1], \quad x \mapsto \cos(x),$   
4c)  $f_3 : \mathbb{R} \rightarrow \mathbb{R}, \quad x \mapsto e^x,$   
4d)  $f_4 : [-1, 1] \rightarrow [0, 1], \quad x \mapsto |x|,$   
4e)  $f_5 : [0, \pi] \rightarrow [0, 1], \quad x \mapsto \sin(x),$   
4f)  $f_6 : [0, \pi] \rightarrow [-1, 1], \quad x \mapsto \cos(x).$

**MC-Problem 5** We are given the matrix

$$\mathbf{A} = \begin{bmatrix} 3 & -2 \\ 5 & -3 \end{bmatrix} \in \mathbb{C}^{2 \times 2}.$$

We are informed that  $\lambda = i$  is an eigenvalue of  $\mathbf{A}$ .

Which of the vectors below is an eigenvector corresponding to the eigenvalue  $\lambda = i$ ?

$$5a) \begin{bmatrix} 3 \\ 3-i \end{bmatrix}, \quad 5b) \begin{bmatrix} 2 \\ 3-i \end{bmatrix}, \quad 5c) \begin{bmatrix} 1 \\ 2+i \end{bmatrix},$$

$$5d) \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad 5e) \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad 5f) \begin{bmatrix} 2 \\ 3+i \end{bmatrix}, \quad 5g) \begin{bmatrix} 1+i \\ 1+i \end{bmatrix}.$$

**MC-Problem 6** We are given the following system of differential equations:

$$\begin{bmatrix} f_1'(t) \\ f_2'(t) \end{bmatrix} = \mathbf{A} \cdot \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}, t \in \mathbb{R} \quad \text{where} \quad \mathbf{A} = \begin{bmatrix} -1 & 2 \\ 0 & -2 \end{bmatrix}.$$

We are informed that the desired solution satisfies the initial-value condition

$$\begin{bmatrix} f_1(0) \\ f_2(0) \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

Which of the expressions below is a solution to the system of differential equations with the given initial value?

$$6a) \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \quad 6b) \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} = \begin{bmatrix} -8e^{-2t} \\ 4e^{-2t} \end{bmatrix},$$

$$6c) \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} = \begin{bmatrix} e^{-t} + e^{-2t} \\ e^{-2t} \end{bmatrix}, \quad 6d) \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} = \begin{bmatrix} 11e^{-t} - 8e^{-2t} \\ 4e^{-2t} \end{bmatrix},$$

$$6e) \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} = \begin{bmatrix} 2e^{-t} + e^{2t} \\ 2e^{-t} + 2e^{2t} \end{bmatrix}, \quad 6f) \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} = \begin{bmatrix} 2e^t + e^{-2t} \\ 2e^t + 2e^{-2t} \end{bmatrix},$$

$$6g) \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} = \begin{bmatrix} 3e^t \\ e^{-t} + 3e^{2t} \end{bmatrix}.$$

**MC-Problem 7** A function  $f : \mathbb{N} \rightarrow \mathbb{R}$  is given recursively by

$$f(1) = 0, \quad f(2) = 1, \quad f(n) = 2f(n-1) + (f(n-2))^2, \quad n \geq 2.$$

Which of the numbers below equals  $f(5)$ ?

- 7a) 12    7b) 13    7c) 14    7d) 15    7e) 16    7f) 17    7g) 18

**MC-Problem 8** Let  $L : V \rightarrow W$  be a linear map from the vector space of polynomials of degree no higher than 2 to the vector space of polynomials of degree no higher than 1. Let an ordered basis for  $V$  be  $\beta = (1, Z, Z^2)$  and an ordered basis for  $W$  be  $\gamma = (1, Z)$ . The mapping matrix of  $L$  is given by

$$\gamma[L]\beta = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & -1 \end{bmatrix}.$$

Which of the polynomials below is equal to  $L(1 + Z + Z^2)$ ?

- 8a)  $1 + Z$     8b)  $Z$     8c)  $1$     8d)  $4Z$     8e)  $9$     8f)  $2 + 3Z$     8g)  $-1 - 3Z$

---

END OF THE MULTIPLE-CHOICE PART. THE SECOND PART STARTS ON  
THE NEXT PAGE

---

**Problem 9** We are given the complex numbers  $z_1 = -2 + 2i$  and  $z_2 = 4 + 5i$ .

- a) Write  $z_2/z_1$  in rectangular form and write  $z_1^4$  in polar form.

**Problem 10** We are given the following real linear system of equations in the four unknowns  $x_1, x_2, x_3$ , and  $x_4$ :

$$\begin{cases} 3x_1 + 2x_2 + x_3 = 0 \\ 2x_1 + x_2 + x_4 = 0 \\ x_3 - x_4 = 1 \end{cases}$$

- a) Determine the general solution to this system of equations.

**Problem 11**

- a) Is the logical proposition  $(P \Leftrightarrow \neg Q) \wedge P$  logically equivalent to the logical proposition  $P \wedge (\neg Q)$ ?
- b) We define the set  $S = (\mathbb{R} \setminus \mathbb{N}) \cap \mathbb{Z}$ . Is the number  $-2$  an element of  $S$ ?

**Problem 12** The following vectors in  $\mathbb{R}^4$  are given:

$$\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 2 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 3 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} -3 \\ -1 \\ -1 \\ 4 \end{bmatrix}, \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} 2 \\ 1 \\ 4 \\ 3 \end{bmatrix}.$$

- a) Find an ordered basis for  $\text{span}_{\mathbb{R}}(\mathbf{u}, \mathbf{v}, \mathbf{w})$ .
- b) Is the given vector  $\mathbf{x}$  in  $\text{span}_{\mathbb{R}}(\mathbf{u}, \mathbf{v}, \mathbf{w})$ ?

**Problem 13** We are given the following matrix:  $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix} \in \mathbb{R}^{2 \times 2}$ .

- a) Find a diagonal matrix  $\mathbf{D}$  and an invertible matrix  $\mathbf{Q}$  such that  $\mathbf{D} = \mathbf{Q}^{-1}\mathbf{A}\mathbf{Q}$ .

**Problem 14** Let  $V = \mathbb{R}^3$ , and let  $\varepsilon$  be the ordered standard basis for  $V$ . Furthermore, the following ordered basis is chosen for  $V$ :

$$\beta = \left( \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right).$$

We are given the following mapping matrix of a linear map  $L : V \rightarrow V$ :

$${}_{\beta}[L]_{\beta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

- Determine the change-of-basis matrices  ${}_{\varepsilon}[\text{id}_{\mathbb{R}^3}]_{\beta}$  and  ${}_{\beta}[\text{id}_{\mathbb{R}^3}]_{\varepsilon}$ .
- Compute the mapping matrix  ${}_{\varepsilon}[L]_{\varepsilon}$ .

**Problem 15** A function  $f : \mathbb{N} \rightarrow \mathbb{R}$  is defined recursively as follows:

$$f(n) = \begin{cases} 1 & \text{if } n = 1, \\ 2 \cdot f(n-1) + n - 1 & \text{if } n \geq 2. \end{cases}$$

- Show by induction on  $n$  that  $f(n) \geq 2^n$  for all  $n \in \mathbb{Z}_{\geq 3}$ .

END OF THE EXAM SHEET