The Technical University of Denmark

Written exam, May 26, 2026

Course name: Mathematics 1a (polytechnical foundation) Course no. 01003
Exam duration: 4 hours

Aids: Written materials permitted (no pocket calculator permitted)

Weight: The problem sheet consists of two parts: Problems 1-5 where answers are to be accompanied by reasoning, and a multiple-choice (MC) part covering MC-Problems 1-8.

The two parts of this exam weigh equally. The individual MC-Problems in the MC part weigh equally; a wrong or missing answer gives 0 points. In the first part, all sub-questions weigh equally.

Further information: Only answers should be provided for the MC problems. In the other part, all answers should be well-reasoned, and intermediate calculations should be provided to an appropriate extent.

Problem 1

A function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is defined by the expression $f(x) = x^2 - x$.

a) Calculate $f(0)$, $f(1)$ and $f(2)$.

b) Is the function f injective?

c) Is the function f invertible?

Problem 2

A complex number z is given in polar form as $z = 2e^{i\pi/6}$.

a) Determine z in rectangular form.

b) Determine z^6 in rectangular form.

c) State a positive integer n such that z^n is a positive real number.

Problem 3

We are given the matrix

$$\mathbf{C} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \in \mathbb{R}^{3\times 3}.$$

a) Provide an ordered basis for the column space $\text{colsp}_{\mathbb{R}}(\mathbf{C})$, and determine the rank of $\mathbf{C}$.

b) Find the determinant of $\mathbf{C}$.

Problem 4

Let $V = \mathbb{R}^{2\times 2}$ denote the real vector space consisting of all 2×2 matrices with real entries. We are given the following two matrices in V:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}.$$

Let $\mathbf{0}$ denote the zero matrix in V. We are given a map $L : V \to V$ defined by the expression $L(\mathbf{M}) = \mathbf{A} \cdot \mathbf{M} - \mathbf{M} \cdot \mathbf{A}$.

a) Calculate $L(\mathbf{0})$, $L(\mathbf{A})$ and $L(\mathbf{B})$.

b) Justify that L is a linear map.

c) We are given the ordered basis

$$\varepsilon = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right)$$

for V. Calculate the mapping matrix ${}_{\varepsilon}[L]_{\varepsilon}$.

Problem 5

We are given the following matrix:

$$\mathbf{M} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \in \mathbb{R}^{2\times 2}.$$

a) Find a diagonal matrix $\mathbf{D}$ and an invertible matrix $\mathbf{Q}$ such that $\mathbf{D} = \mathbf{Q}^{-1}\mathbf{M}\mathbf{Q}$.

b) Show by induction on n that

$$\mathbf{M}^n = \begin{bmatrix} 1 & 2^n - 1 \\ 0 & 2^n \end{bmatrix}$$

for all natural numbers n.

MC Part – hand in this sheet along with your solution to the previous problems.

Student Number: _____

MC Problem 1. Let P and Q be logical propositions.
Which of the following logical propositions is equivalent to

$$(\neg Q \wedge P) \wedge \neg(\neg P \vee Q)\,?$$

Answer Options (mark only one):

◯ $P \vee \neg Q$	◯ $P \wedge Q$	◯ $\neg P \wedge Q$	◯ $\neg P \wedge \neg Q$
◯ $P \wedge \neg Q$	◯ $\neg P \vee Q$	◯ $P \Rightarrow Q$	◯ $Q \Rightarrow P$

MC Problem 2. What is the polar form of the complex number $z = 1 - i$?

Answer Options (mark only one):

◯ $\sqrt{2}\,e^{-i\pi/4}$	◯ $2e^{-i\pi/4}$	◯ $\sqrt{2}\,e^{i\pi/4}$	◯ $2e^{i\pi/4}$
◯ $e^{-i\pi/2}$	◯ $\sqrt{2}\,e^{-i\pi/2}$	◯ $\sqrt{2}\,e^{i\pi/2}$	◯ $e^{-i\pi/4}$

MC Problem 3. Which of the below polynomials is the result of the division

$$\frac{2Z^3 + Z^2 - 12Z + 9}{2Z - 3}\,?$$

Answer Options (mark only one):

◯ $Z^2 + 2Z - 1$	◯ $Z^2 - Z + 3$	◯ $Z^2 + 2Z - 3$	◯ $Z^2 - 2Z + 3$
◯ $Z^2 + Z - 3$	◯ $Z^2 - 3Z + 1$	◯ $Z^2 + 3Z - 1$	◯ $Z^2 - Z - 3$

MC Problem 4. Let

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 1 & 3 \end{bmatrix}.$$

Choose the result of the calculation $\mathbf{A}^{-1} \cdot \mathbf{b}^T$.

Answer Options (mark only one):

◯ $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$	◯ $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$	◯ $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$	◯ $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$
◯ $\begin{bmatrix} 1 & -2 \end{bmatrix}$	◯ $\begin{bmatrix} 2 & 1 \end{bmatrix}$	◯ $\begin{bmatrix} -1 & -2 \end{bmatrix}$	◯ $\begin{bmatrix} 3 & 1 \end{bmatrix}$

MC Problem 5. Which of the following numbers is an eigenvalue of the matrix

$$\begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix}\,?$$

Answer Options (mark only one):

◯ -4	◯ -1	◯ 0	◯ 1
◯ -2	◯ 3	◯ 4	◯ 6

MC Problem 6. Let $f : \mathbb{N} \to \mathbb{R}^2$ be a recursively defined function given by

$$f(1) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \qquad f(n) = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} f(n-1),\ n \geq 2.$$

Which of the below vectors equals $f(3)$?

Answer Options (mark only one):

◯ $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$	◯ $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$	◯ $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$	◯ $\begin{bmatrix} 1 \\ -4 \end{bmatrix}$
◯ $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$	◯ $\begin{bmatrix} 0 \\ -4 \end{bmatrix}$	◯ $\begin{bmatrix} 1 \\ 7 \end{bmatrix}$	◯ $\begin{bmatrix} 1 \\ -7 \end{bmatrix}$

MC Problem 7. Let V be a subspace of $\mathbb{R}^3$ with ordered basis

$$b = \left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right).$$

Which of the following vectors in $\mathbb{R}^3$ does not belong to V?

Answer Options (mark only one):

◯ $\begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$	◯ $\begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$	◯ $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$	◯ $\begin{bmatrix} 4 \\ 2 \\ 4 \end{bmatrix}$
◯ $\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$	◯ $\begin{bmatrix} 5 \\ -1 \\ 5 \end{bmatrix}$	◯ $\begin{bmatrix} -2 \\ 0 \\ -2 \end{bmatrix}$	◯ $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

MC Problem 8. Let $a, b \in \mathbb{R}$ be constants and let $q : \mathbb{R} \to \mathbb{R}$ be a function.
Consider the differential equation

$$f''(t) + af'(t) + bf(t) = q(t)\,.$$

For which values of a, b and $q(t)$ is the function

$$f(t) = e^t \cos(2t) + 3e^t \sin(2t) + 5t$$

a solution?

Answer Options (mark only one):

◯ $a = -2, \quad b = 5, \quad q(t) = 10t - 25$	◯ $a = -2, \quad b = 2, \quad q(t) = 10t - 10$
◯ $a = 2, \quad b = 5, \quad q(t) = 25t + 10$	◯ $a = -1, \quad b = 5, \quad q(t) = 25t$
◯ $a = -2, \quad b = 4, \quad q(t) = 20t - 10$	◯ $a = 2, \quad b = 5, \quad q(t) = 25t - 10$
◯ $a = -2, \quad b = 5, \quad q(t) = 25t - 10$	◯ $a = -2, \quad b = 3, \quad q(t) = 15t - 10$

END OF EXAM