1MC

Expanding from column 1:

$$\det(\underline{A}) = -(-1)\det\begin{bmatrix}1&2\\1&2\end{bmatrix} + 1\cdot\det\begin{bmatrix}1&2\\4&1\end{bmatrix} = -7 \quad (1e)$$

2MC

$$3f''(t) - 6f'(t) + 15f(t) = 0 \iff f''(t) - 2f'(t) + 5f(t) = 0$$

Characteristic polynomial:

$$p(\lambda) = \lambda^2 - 2\lambda + 5 \qquad p(\lambda) = 0: \ \lambda^2 - 2\lambda + 5 = 0$$

$$\lambda_1 = 1+2i, \ \lambda_2 = 1-2i$$

$$f(t) = c_1 e^t \cos 2t + c_2 e^t \sin 2t \quad (2b)$$

3MC

$$\begin{array}{l} z+1 \,|\, z^3 - 2z^2 - z + 7 \,|\, \underline{z^2 - 3z + 2} \\ \quad z^3 + z^2 \\ \hline \quad\quad -3z^2 - z \\ \quad\quad -3z^2 - 3z \\ \hline \quad\quad\quad 2z + 7 \\ \quad\quad\quad 2z + 2 \\ \hline \quad\quad\quad\quad 5 \end{array} \quad (3d)$$

Alternatively:

$$P(z) = d(z)\cdot q(z) + r(z) \qquad d(z) = z+1$$

$$P(-1) = 0 \cdot q(z) + r(z) = 5$$

4MC

4a) Neither surjective or injective

4b) Not injective

4c) Not surjective

4d) Not injective

4e) Not injective

4f) Bijective (correct)

5MC

Eigenvectors:

$$\begin{bmatrix} 3-i & -2 \\ 5 & -3-i \end{bmatrix} \rightarrow \begin{bmatrix} 3-i & -2 \\ 0 & 0 \end{bmatrix} \qquad \begin{matrix} x_1 = t \\ x_2 = \frac{(3-i)}{2} \cdot t \end{matrix}$$

$$\underline{v} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3-i \end{pmatrix} \ (5b)$$

6MC

Eigenvalues of $\underline{A}$: $\lambda_1 = -1$, $\lambda_2 = -2$

Corresponding eigenvectors: $\underline{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\underline{v}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

General solution:

$$\begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-2t} \quad , \ c_1, c_2 \in \mathbb{C}$$

$$\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} f_1(0) \\ f_2(0) \end{bmatrix} = \begin{bmatrix} c_1 - 2c_2 \\ c_2 \end{bmatrix} \Rightarrow \begin{matrix} c_2 = 4 \\ c_1 = 11 \end{matrix}$$

$$\begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} = \begin{bmatrix} 11e^{-t} - 8e^{-2t} \\ 4e^{-2t} \end{bmatrix} \ (6d)$$

7MC

$f(1) = 0$, $f(2) = 1$, $f(3) = 2f(2) + (f(1))^2 = 2$

$f(4) = 2f(3) + (f(2))^2 = 5$

$f(5) = 2f(4) + (f(3))^2 = 14$ (7c)

8MC

$$[L(1+z+z^2)]_\gamma = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

$$L(1+z+z^2) = 0 \cdot 1 + 4 \cdot z = 4z \ (8d)$$

(9)

$$\frac{z_2}{z_1} = \frac{(4+5i)(-2-2i)}{(-2+2i)(-2-2i)} = \frac{-8+10+(-8-10)i}{8} = \underline{\underline{\frac{1}{4} - \frac{9}{4}i}}$$

$$z_1 = \sqrt{8} \cdot e^{i\frac{3\pi}{4}}$$

$$(z_1)^4 = 64\, e^{i\left(\frac{3\pi}{4}\right)\cdot 4} = 64 e^{i3\pi} = \underline{\underline{64\, e^{i\pi}}}$$

(10) Augmented matrix:

$$\underline{\underline{I}} = \left[\begin{array}{cccc|c} 3 & 2 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & 1 & -1 & 1 \end{array}\right]$$

$x_4 = t$, $x_3 = 1+t$, $x_2 = -2+t$, $x_1 = 1-t$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad t \in \mathbb{R}}}$$

(11) Truth table:

a)

P	Q	$\neg Q$	$P \Leftrightarrow \neg Q$	$(P \Leftrightarrow \neg Q) \wedge P$	$P \wedge (\neg Q)$
T	T	F	F	F	F
F	T	F	T	F	F
T	F	T	T	T	T
F	F	T	F	F	F

The two propositions are logically equivalent.

b) $-2 \in \mathbb{R} \wedge -2 \notin \mathbb{N} \Rightarrow -2 \in \mathbb{R} \setminus \mathbb{N}$

$-2 \in \mathbb{Z}$

$\Downarrow$

$-2 \in S = (\mathbb{R} \setminus \mathbb{N}) \cap \mathbb{Z}$

(12)

$$\underline{\underline{M}} = \begin{bmatrix} | & | & | & | \\ \underline{u} & \underline{v} & \underline{w} & \underline{x} \\ | & | & | & | \end{bmatrix} = \left[\begin{array}{ccc|c} 1 & 3 & -3 & 2 \\ 1 & 2 & -1 & 1 \\ -1 & -1 & -1 & 4 \\ 2 & 1 & 4 & 3 \end{array}\right] \xrightarrow[\text{rref}]{} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

a) An ordered basis for $\text{span}_{\mathbb{R}}(\underline{u}, \underline{v}, \underline{w})$ could be $(\underline{u}, \underline{v})$

b) We see that $\underline{x}$ is not in $\text{span}_{\mathbb{R}}(\underline{u}, \underline{v}, \underline{w})$

(13)

We have $\underline{\underline{D}} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, $\underline{\underline{Q}} = \begin{bmatrix} | & | \\ \underline{v}_1 & \underline{v}_2 \\ | & | \end{bmatrix}$

where λ_i is an eigenvalue of $\underline{\underline{A}}$ with eigenvector $\underline{v}_i$

Finding eigenvalues:

We see directly that $\lambda_1 = 1$, $\lambda_2 = 3$.

Finding eigenvectors:

$\underline{v}_1$: $\begin{bmatrix} 1-1 & 0 \\ 1 & 3-1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}$ $\underline{v}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$\underline{v}_2$: $\begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix}$ $\underline{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\underline{\underline{D}} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$, $\underline{\underline{Q}} = \begin{bmatrix} -2 & 0 \\ 1 & 1 \end{bmatrix}$

(14)

a) ${}_{\mathcal{E}}[id_{\mathbb{R}^3}]_{\beta}$ can be written out directly:

$$ {}_{\mathcal{E}}[id_{\mathbb{R}^3}]_{\beta} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

$$ {}_{\beta}[id_{\mathbb{R}^3}]_{\mathcal{E}} = {}_{\mathcal{E}}[id_{\mathbb{R}^3}]_{\beta}^{-1} : $$

$$ \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] $$

$$ {}_{\beta}[id_{\mathbb{R}^3}]_{\mathcal{E}} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

b)

$$ {}_{\mathcal{E}}[L]_{\mathcal{E}} = {}_{\mathcal{E}}[id_{\mathbb{R}^3}]_{\beta}\ {}_{\beta}[L]_{\beta}\ {}_{\beta}[id_{\mathbb{R}^3}]_{\mathcal{E}} $$

$$ = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

$$ = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & -1 & 0 \\ -3 & 3 & 0 \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 & 0 \\ -4 & 5 & 0 \\ -1 & 1 & 1 \end{bmatrix} $$

(15)

Using thm 3.5.1 in the textbook.

1. Base case $(n=3)$

$f(3) = 2f(2) + 3 - 1 = 2(2\cdot 1 + 2 - 1) + 2 = 8 \geq 2^3$ ok

2. Induction step: $(n > 3)$

Assuming true for $n-1$, then showing true for n.

Induction hypothesis: $f(n-1) \geqslant 2^{n-1}$

$$f(n) = 2 \cdot f(n-1) + n - 1 \geqslant 2 \cdot 2^{n-1} + n - 1$$
$$= 2^n + n - 1 \geqslant 2^n \quad (\text{as } n > 3)$$

So, $f(n) \geqslant 2^n$ is shown.

Theorem 3.5.1 then tells that the proposition is true for all $n \in \mathbb{Z}_{\geqslant 3}$