Technical University of Denmark

Written test exam, November 2023

Course name: Mathematics 1a (Polytechnical foundation) Course nr. 01003
Exam duration: 2 hours

Aid: No electronic aid

“Weighting”: All questions in this exam are weighted equally. This part of the test exam constitutes 50% of the entire test exam.

Additional information: The questions are posed first in English and then in Danish. All answers have to be motivated and intermediate steps need to be given to a reasonable extent.

Question 1

Are the logical propositions $(\neg(P \vee Q)) \Rightarrow P$ and $P \vee Q$ logically equivalent?

Question 2

Compute the roots of the polynomial $Z^3 + 27$. The roots should be given in rectangular form.

Question 3

A sequence of numbers $(s_1, s_2, s_3, \dots)$ is defined recursively in the following way:

$$s_n = \begin{cases} 0 & \text{if } n = 1, \\ 2s_{n-1} + 2 & \text{if } n \geq 2. \end{cases}$$

a) Compute s_1, s_2 and s_3.

b) Show using induction on n that $s_n = 2^n - 2$ for all $n \in \mathbb{Z}_{\geq 1}$.

Question 4

Given is the following system of linear equations in the indeterminates x_1, x_2, x_3, x_4 over $\mathbb{R}$:

$$\begin{cases} x_1 + x_2 + 2x_3 &= 1 \\ x_2 - 4x_3 + 4x_4 &= 5 \\ 3x_1 + 3x_2 + 2x_3 + 4x_4 &= -1 \end{cases}.$$

a) Is the given system of linear equations homogeneous or inhomogeneous?

b) Let $\mathbf{v} = (v_1, v_2, v_3, v_4) \in \mathbb{R}^4$ and $\mathbf{w} = (w_1, w_2, w_3, w_4) \in \mathbb{R}^4$ be two solutions to the given system of linear equations. Is $\mathbf{v} - \mathbf{w}$ also a solution to the system?

Question 5

Given is the following matrix:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 0 & 1 \\ 3 & 0 & 8 \end{bmatrix} \in \mathbb{C}^{3\times 3}.$$

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a) Show that the characteristic polynomial of the given matrix **A** is given by $p_{\mathbf{A}}(Z) = -Z^3 + 9Z^2 - 8Z$.

b) You are given that the vectors

$$\begin{bmatrix} -7 \\ -25 \\ 3 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 1 \\ 8 \end{bmatrix}$$

are eigenvectors of the matrix **A**. What are the corresponding eigenvalues?

Question 6

Let V be a real vectorspace of dimension two. It is given that for two ordered bases β and γ of V, the change of basis matrix ${}_{\gamma}[\mathrm{id}]_{\beta}$ equals:

$${}_{\gamma}[\mathrm{id}]_{\beta} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}.$$

Compute the matrix ${}_{\beta}[\mathrm{id}]_{\gamma}$.

Question 7

Given is the following differential equation: $f''(t) - 3f'(t) + 2f(t) = 2t$.

a) Check that the function $f(t) = t + 3/2$ is a solution to the given differential equation.

b) Compute the general solution of the given differential equation.

END OF THE EXAM

Opgave 1

Er de logiske udsagn $(\neg(P \vee Q)) \Rightarrow P$ og $P \vee Q$ logisk ækvivalente?

Opgave 2

Beregn rødderne i polynomiet $Z^3 + 27$. Rødderne ønskes angivet på rektangulær form.

Opgave 3

En følge af tal $(s_1, s_2, s_3, \dots)$ defineres rekursivt på følgende måde:

$$s_n = \begin{cases} 0 & \text{hvis } n = 1, \\ 2s_{n-1} + 2 & \text{hvis } n \geq 2. \end{cases}$$

a) Bestem s_1, s_2 og s_3.

b) Vis ved hjælp af induktion efter n at $s_n = 2^n - 2$ for alle $n \in \mathbb{Z}_{\geq 1}$.

Opgave 4

Givet følgende lineære ligningsystem over $\mathbb{R}$ i de ubekendte x_1, x_2, x_3, x_4:

$$\begin{cases} x_1 + x_2 + 2x_3 &= 1 \\ x_2 - 4x_3 + 4x_4 &= 5 \\ 3x_1 + 3x_2 + 2x_3 + 4x_4 &= -1 \end{cases}.$$

a) Er det givne lineære ligningssystem homogent eller inhomogent?

b) Lad $\mathbf{v} = (v_1, v_2, v_3, v_4) \in \mathbb{R}^4$ og $\mathbf{w} = (w_1, w_2, w_3, w_4) \in \mathbb{R}^4$ være to løsninger til det givne lineære ligningssystem. Er $\mathbf{v} - \mathbf{w}$ også løsning til systemet?

Opgave 5

Givet følgende matrix:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 0 & 1 \\ 3 & 0 & 8 \end{bmatrix} \in \mathbb{C}^{3\times 3}.$$

a) Vis at det karakteristiske polynomium af matricen $\mathbf{A}$ er $p_{\mathbf{A}}(Z) = -Z^3 + 9Z^2 - 8Z$.

b) Det oplyses at vektorerne

$$\begin{bmatrix} -7 \\ -25 \\ 3 \end{bmatrix} \quad \text{og} \quad \begin{bmatrix} 0 \\ 1 \\ 8 \end{bmatrix}$$

er egenvektorer af matricen **A**. Hvad er deres tilhørende egenværdier?

Opgave 6

Lad V være et reelt vektorrum af dimension to. Det oplyses at for to ordnede baser β og γ for V opfylder basisskiftematricen ${}_\gamma[\mathrm{id}]_\beta$:

$${}_\gamma[\mathrm{id}]_\beta = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}.$$

Bestem nu matricen ${}_\beta[\mathrm{id}]_\gamma$.

Opgave 7

Givet følgende differentialligning: $f''(t) - 3f'(t) + 2f(t) = 2t$.

a) Vis at funktionen $f(t) = t + 3/2$ er løsning til den givne differentialligning.

b) Beregn differentialligningens fuldstændige løsning.

EKSAMEN SLUT