Exam 01003 E24

Der anvendes en scoringsalgoritme, som er baseret på "One best answer"

Dette betyder følgende:
Der er altid netop ét svar som er mere rigtigt end de andre
Studerende kan kun vælge ét svar per spørgsmål
Hvert rigtigt svar giver 1 point
Hvert forkert svar giver 0 point (der benyttes IKKE negative point)

The following approach to scoring responses is implemented and is based on "One best answer"
There is always only one correct answer – a response that is more correct than the rest
Students are only able to select one answer per question
Every correct answer corresponds to 1 point
Every incorrect answer corresponds to 0 points (incorrect answers do not result in subtraction of points)

We are given the quadratic equation:

$$3z^2 - 6z + 12 = 0.$$

Which of the following complex numbers is a solution to the equation?

Vælg en svarmulighed

- ○ $z = 3e^{\frac{\pi}{2}i}$
- ○ $z = 2e^{\pi i}$
- ☒ $z = 2e^{-\frac{\pi}{3}i}$
- ○ $z = 3e^{\frac{\pi}{3}i}$
- ○ $z = 12e^{-\frac{\pi}{2}i}$
- ○ None of the shown answers is a solution to the equation.
- ○ $z = -6e^{12i}$

$$z = \frac{-(-6) \pm \sqrt{(-6)^2 - 4 \cdot 3 \cdot 12}}{2 \cdot 3} = 1 \pm \frac{\sqrt{-108}}{6}$$

$$= 1 \pm i\sqrt{\frac{108}{36}} = 1 \pm i\sqrt{3}$$

<u>Modulus</u>

$$|1 \pm i\sqrt{3}| = \sqrt{1^2 + \sqrt{3}^2} = 2$$

<u>Arguments</u>

$$\sin(\theta) = \frac{\sqrt{3}}{2}$$

so arguments are

$$\pm \frac{\pi}{3}$$

Polar forms: $2e^{\frac{\pi}{3}i}$, $\underline{\underline{2e^{-\frac{\pi}{3}i}}}$

Let $p(Z)$ be a polynomial of degree 4.

$$p(Z) = a_0 + a_1 Z + a_2 Z^2 + a_3 Z^3 + a_4 Z^4$$

We are given the following information:
1) All of the coefficients a_0, a_1, a_2, a_3, a_4 are real.
2) 2 is a root of $p(Z)$ with a multiplicity of 2.
3) $1 + i$ is a root of $p(Z)$.
4) $p(1) = 7$.

What are the values of a_0 and a_3?

Vælg en svarmulighed

- ○ None of the provided answer options are correct.
- ○ $a_0 = 2, a_3 = 7$
- ○ $a_0 = 2, a_3 = 1 + i$
- ○ $a_0 = -77, a_3 = 33$
- ☒ $a_0 = 56, a_3 = -42$
- ○ $a_0 = 7, a_3 = -6$
- ○ $a_0 = -6, a_3 = 8$

Since $p(z)$ is real and $1+i$ is a root, then also $1-i$ is a root. All roots: $2, 2, 1+i, 1-i$.

Factorized form:

$$p(z) = a_4 (z-2)^2 (z-1+i)(z-1-i)$$

We have:

$$p(1) = a_4 (-1)^2 \cdot i \cdot (-i) = a_4 = 7$$

Hence:

$$\begin{aligned} p(z) &= 7(z-2)^2(z-1+i)(z-1-i) \\ &= 7(z^2+4-4z)(z^2-2z+1-i^2) \\ &= 7(z^4-2z^3+2z^2+4z^2-8z+8-4z^3+8z^2-8z) \\ &= 7z^4 \underline{-42} z^3 + 98z^2 - 112z + \underline{56} \end{aligned}$$

Let the function $f : \mathbb{N} \to \mathbb{N}$ be given by:

$$f(n) = \begin{cases} 1 & \text{for } n = 1 \\ 2 & \text{for } n = 2 \\ 3f(n-1) - f(n-2) & \text{for } n \geq 3 \end{cases}$$

What is the value of $f(5)$?

Vælg en svarmulighed

- $f(5) = 13$
- $f(5) = 15$
- $f(5) = 5$
- $f(5) = 1$
- None of the provided answer options are correct.
- $f(5) = 26$
- ☒ $f(5) = 34$

$f(1) = 1$, $f(2) = 2$, $f(3) = 3 \cdot f(2) - f(1) = 5$

$f(4) = 3\,f(3) - f(2) = 3 \cdot 5 - 2 = 13$

$f(5) = 3\,f(4) - f(3) = 3 \cdot 13 - 5 = \underline{\underline{34}}$

Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear map of the form:

$$L(\mathbf{v}) = \mathbf{A}\mathbf{v}, \text{ where } \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \text{ is a real } 3 \times 3 \text{ matrix.}$$

We are being informed that:

$$L(\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}) = -2 \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \quad , \quad L(\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}) = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \quad ,$$

$$L(\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}) = 3 \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

Which of the below options correct state two of the elements in the matrix $\mathbf{A}$?

Vælg en svarmulighed

- ○ $a_{11} = \frac{1}{2}$, $a_{32} = \frac{3}{2}$
- ○ $a_{11} = \frac{5}{2}$, $a_{32} = 2$
- ☒ $a_{11} = -\frac{1}{2}$, $a_{32} = -5$
- ○ $a_{11} = 1$, $a_{32} = 1$
- ○ $a_{11} = -2$, $a_{32} = 3$
- ○ $a_{11} = \frac{17}{2}$, $a_{32} = -\frac{3}{2}$
- ○ None of the provided answer options are correct.

We see eigenvectors

$\underline{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ with $\lambda_1 = -2$

$\underline{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ with $\lambda_2 = 1$

$\underline{v}_3 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ with $\lambda_3 = 3$

All belong to different eigenspaces, so linearly indep.
So, a diagonalization is:

$\underline{\underline{V}}^{-1}\underline{\underline{A}}\,\underline{\underline{V}} = \underline{\underline{\Lambda}}$ where $\underline{\underline{V}} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ -1 & 1 & -1 \end{bmatrix}$ and $\underline{\underline{\Lambda}} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

with $\underline{\underline{V}}^{-1} = \begin{bmatrix} 1/2 & -1 & -3/2 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 1 \end{bmatrix}$ we get: $\underline{\underline{A}} = \underline{\underline{V}}\,\underline{\underline{\Lambda}}\,\underline{\underline{V}}^{-1} = \begin{bmatrix} \underline{\underline{-1/2}} & 5 & 13/2 \\ -3/2 & 8 & 17/2 \\ 3/2 & \underline{\underline{-5}} & -11/2 \end{bmatrix}$

Let $p(t) = at + b$, where $a, b \in \mathbb{R}$, be a first-degree polynomial.
Consider a second-order differential equation of the form:

$$f''(t) + 2f'(t) - 8f(t) = p(t)$$

We are being informed that $f_0(t) = -t + 5$ is a particular solution to the differential equation.
Which of the below options states a particular solution to the differential equation, $f_p(t)$, along with the values of a and b?

Vælg en svarmulighed

- $f_p(t) = e^{-2t} - t + 5 \;,\; a = -1 \;,\; b = 5$
- $f_p(t) = 4e^{4t} - t + 5 \;,\; a = 8 \;,\; b = -42$
- $f_p(t) = 2e^{2t} \;,\; a = 8 \;,\; b = -42$
- $f_p(t) = e^{2t} + e^{-4t} - t + 5 \;,\; a = -1 \;,\; b = 5$
- X $f_p(t) = e^{-4t} - t + 5 \;,\; a = 8 \;,\; b = -42$
- $f_p(t) = -t + 5 \;,\; a = -1 \;,\; b = 5$
- None of the provided answer options are correct.

Inserting $f_0(t) = -t+5$:

$\Updownarrow$ $(\cancel{-t+5})'' + 2(-t+5)' - 8(-t+5) = p(t)$

$-2 + 8t - 40 = p(t)$

so $p(t) = \underline{\underline{8}}t - \underline{\underline{42}}$.

Corresponding homogeneous eq:

$f''(t) + 2f'(t) - 8f(t) = 0$

so $\lambda^2 + 2\lambda - 8 = 0 \iff \lambda = \dfrac{-2 \pm \sqrt{4+4\cdot 8}}{2} = -1 \pm 3 = 2 \vee -4$

Sol. to homog. eq is: $f(t) = c_1 e^{2t} + c_2 e^{-4t}$, $c_1, c_2 \in \mathbb{C}$

Choosing $c_1 = 1$, $c_2 = 0$ and adding $f_0(t)$ gives, acc. to the structural theorem, a particular solution to the inhom. eq: $\underline{\underline{f(t) = e^{-4t} - t + 5}}$

We are given the matrix $\mathbf{A} \in \mathbb{R}^{3\times 3}$.

$$\mathbf{A} = \begin{bmatrix} -1 & 4 & 4 \\ 0 & 7 & 8 \\ 0 & -4 & -5 \end{bmatrix}$$

Which of the following vectors is not an egenvector of the matrix?

Vælg en svarmulighed

- [x] $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$
- [ ] $\begin{pmatrix} 2 \\ -3 \\ 3 \end{pmatrix}$
- [ ] $\begin{pmatrix} 3 \\ 6 \\ -3 \end{pmatrix}$
- [ ] $\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$
- [ ] $\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$
- [ ] $\begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$
- [ ] All shown vectors are eigenvectors of matrix $\mathbf{A}$.

Eigenvalues:

choose column $j=1$

$$\det\left(\begin{bmatrix} -1-\lambda & 4 & 4 \\ 0 & 7-\lambda & 8 \\ 0 & -4 & -5-\lambda \end{bmatrix}\right) = \sum_{i=1}^{3} (-1)^{i+1} a_{i1} \det(\underline{A}(i;1))$$

$$= (-1)^2(-1-\lambda)\det\left(\begin{bmatrix} 7-\lambda & 8 \\ -4 & -5-\lambda \end{bmatrix}\right)$$

$$= (-1-\lambda)\left((7-\lambda)(-5-\lambda)+4\cdot 8\right)$$

$$= (-1-\lambda)(-35-2\lambda+\lambda^2+32)$$

$$= (-1-\lambda)(\lambda^2-2\lambda-3)$$

$$\lambda = \frac{2 \pm \sqrt{4+12}}{2} = 1 \pm \frac{\sqrt{16}}{2} = 3 \vee -1$$

So $\lambda_1 = -1$ with $am(-1) = 2$ and $\lambda_2 = 3$

Finding E_{-1}:

$$(\underline{A} - (-1)\underline{\underline{I}}_3)\underline{v}_1 = \underline{0} \Leftrightarrow \left[\begin{array}{ccc|c} 0 & 4 & 4 & 0 \\ 0 & 8 & 8 & 0 \\ 0 & -4 & -4 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

so $\underline{v}_1 = t\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$, $t, s \in \mathbb{R}$, meaning $E_{-1} = \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}\right)$

Finding E_3:

$$\left[\begin{array}{ccc|c} -4 & 4 & 4 & 0 \\ 0 & 4 & 8 & 0 \\ 0 & -4 & -8 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

so $\underline{v}_2 = t\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$, $t \in \mathbb{R}$ and $E_3 = \operatorname{span}\left(\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}\right)$

Among the options, only $\underline{\underline{\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}}}$ is not in either eigenspace.

Let $\mathbf{A} \in \mathbb{C}^{4\times 4}$ be given by:

$$\mathbf{A} = \begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 4 & 3 & 2 \\ 2 & 3 & 4 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

Which of the below numbers states the following determinant:

$D = \det((\mathbf{A}^{\mathbf{T}} + \mathbf{A}^{\mathbf{2}}) \cdot \mathbf{A}^{-\mathbf{1}})$?

Vælg en svarmulighed

- $D = -20$
- None of the provided answer options are correct.
- $D = 161$
- $D = 103$
- $D = 20$
- $D = 203$
- $D = -110$

$\underline{\underline{A}}$ is symmetric, so $\underline{\underline{A}} = \underline{\underline{A}}^T$.

$$D = \det((\underline{\underline{A}}^T + \underline{\underline{A}}^2)\underline{\underline{A}}^{-1}) = \det((\underline{\underline{A}} + \underline{\underline{A}}^2)\underline{\underline{A}}^{-1})$$

$$= \det(\underline{\underline{I}}_4 + \underline{\underline{A}})$$

$$= \det\left(\begin{bmatrix} 5 & 3 & 2 & 1 \\ 3 & 5 & 3 & 2 \\ 2 & 3 & 5 & 3 \\ 1 & 2 & 3 & 5 \end{bmatrix}\right) = -\det\left(\begin{bmatrix} 1 & 2 & 3 & 5 \\ 3 & 5 & 3 & 2 \\ 2 & 3 & 5 & 3 \\ 5 & 3 & 2 & 1 \end{bmatrix}\right)$$

$$= -\det\left(\begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & -1 & -6 & -13 \\ 0 & -1 & -1 & -7 \\ 0 & -7 & -13 & -24 \end{bmatrix}\right) = -\det\left(\begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & -1 & -6 & -13 \\ 0 & 0 & 5 & 6 \\ 0 & 0 & 29 & 67 \end{bmatrix}\right)$$

$$= -\det\left(\begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & -1 & -6 & -13 \\ 0 & 0 & 5 & 6 \\ 0 & 0 & 0 & 67 - \frac{29\cdot 6}{5} \end{bmatrix}\right)$$

$$= -1\cdot(-1)\cdot 5\cdot\left(67 - \frac{29\cdot 6}{5}\right) = 5\cdot 67 - 29\cdot 6$$

$$= 335 - 174 = \underline{\underline{161}}$$

Consider the real system of equations:

$$x_1 + x_2 + 3x_3 + x_4 = 7$$
$$-x_1 + x_3 = -5$$

Which of the below options states the general solution to the system of equations?

Vælg en svarmulighed

- $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ 0 \\ 0 \end{pmatrix} + t_1 \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} + t_2 \begin{pmatrix} 1 \\ -4 \\ 1 \\ 0 \end{pmatrix}, \quad t_1, t_2 \in \mathbb{R}$

- $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 0 \\ 2 \end{pmatrix}$

- $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = t_1 \begin{pmatrix} 5 \\ 2 \\ 0 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} + t_3 \begin{pmatrix} 1 \\ -4 \\ 1 \\ 0 \end{pmatrix}, \quad t_1, t_2, t_3 \in \mathbb{R}$

- None of the provided answer options state the general solution.

- $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = t_1 \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} + t_2 \begin{pmatrix} -1 \\ 4 \\ -1 \\ 0 \end{pmatrix}, \quad t_1, t_2 \in \mathbb{R}$

- $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 7 \\ -5 \\ 0 \\ 0 \end{pmatrix} + t_1 \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} + t_2 \begin{pmatrix} 1 \\ -4 \\ 1 \\ 0 \end{pmatrix}, \quad t_1, t_2 \in \mathbb{R}$

- $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ 0 \\ 1 \end{pmatrix} + t_1 \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \quad t_1 \in \mathbb{R}$

$$\left[\begin{array}{cccc|c} 1 & 1 & 3 & 1 & 7 \\ -1 & 0 & 1 & 0 & -5 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 5 \\ 0 & 1 & 4 & 1 & 2 \end{array}\right]$$

$$\Updownarrow \quad \begin{cases} x_1 - x_3 = 5 \\ x_2 + 4x_3 + x_4 = 2 \end{cases} \iff \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -4 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \quad t, s \in \mathbb{R}$$