Solution sketch Math 1a spring 25 reexam (1)

Q1

a)

P	Q	R	$P \vee Q \vee R$	$(P \vee Q \vee R) \Rightarrow Q$
0	0	0	0	1
1	0	0	1	0
0	1	0	1	1
1	1	0	1	1
0	0	1	1	0
1	0	1	1	0
0	1	1	1	1
1	1	1	1	1

b) $A \cup B = \{0, 1, 2, 3\}$

$(A \cup B) \cap C = \{3\}$

Q2

$z^3 = -i \iff z^3 = e^{-i\frac{\pi}{2}}$

$z = re^{iv}$

$r^3 = 1 \iff r = 1 \quad \wedge \quad 3v = -\frac{\pi}{2} + p \cdot 2\pi \iff v = -\frac{\pi}{6} + p \cdot \frac{2\pi}{3}$

$$z_1 = e^{-i\frac{\pi}{6}}, \quad z_2 = e^{i\frac{\pi}{2}}, \quad z_3 = e^{-i\frac{5}{6}\pi}$$

Q3

a) $$\underline{\underline{A}}^2 = \begin{bmatrix} a & 1 \\ 0 & 2a \end{bmatrix}^2 = \begin{bmatrix} a & 1 \\ 0 & 2a \end{bmatrix} \cdot \begin{bmatrix} a & 1 \\ 0 & 2a \end{bmatrix} = \begin{bmatrix} a^2 & 3a \\ 0 & 4a^2 \end{bmatrix}$$

$$= \begin{bmatrix} a^2 & (2^2-1)a^{2-1} \\ 0 & (2a)^2 \end{bmatrix}.$$

Induktion basis !

b) Indution basis ($n=2$) Clear from Side ①.

Assume true for $n-1$.

$$\underline{\underline{A}}^{n-1} = \begin{bmatrix} a^{n-1} & (2^{n-1}-1)a^{n-2} \\ 0 & (2a)^{n-1} \end{bmatrix}$$

$$\underline{\underline{A}}^{n} = \underline{\underline{A}}^{n-1} \cdot \underline{\underline{A}} = \begin{bmatrix} a^{n-1} & (2^{n-1}-1)a^{n-2} \\ 0 & (2a)^{n-1} \end{bmatrix} \begin{bmatrix} a & 1 \\ 0 & 2a \end{bmatrix}$$

$$= \begin{bmatrix} a \cdot a^{n-1} & a^{n-1} + (2^{n-1}-1)a^{n-2} \cdot 2a \\ 0 & (2a)^{n-1} \cdot 2a \end{bmatrix}$$

$$= \begin{bmatrix} a^{n} & (2^{n}-1)a^{n-1} \\ 0 & (2a)^{n} \end{bmatrix}$$

$\blacksquare$

Q4

$$\underline{\underline{B}} = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 2 & -4 \\ 3 & 2 & 4 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

a) $\text{Rank}(\underline{B}) = 2$ (2 pivots)

b) $\text{Null}(\underline{B}) = 1$ (Rank-Nullity theorem, $3 = \text{Rank}(\underline{B}) + \text{Null}(\underline{B})$)

Q 5.

a) Homogeneous.

b) It is not a solution, first equation.

$f_1'(t) = f_1(t) + f_2(t)$, $f_1(t) = e^t$, $f_2(t) = e^{2t}$

$\Rightarrow e^t \stackrel{?}{=} e^t + e^{2t} \Leftrightarrow e^{2t} = 0$. False

c) $\underline{f}'(t) = \underline{\underline{A}}\, \underline{f}(t)$ $\quad \underline{\underline{A}} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$

Eigenvalues and eigenvectors:

$\lambda_1 = 1 \quad \underline{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\lambda_2 = 2 \quad \underline{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}$$

$$= \begin{bmatrix} c_1 e^t + c_2 e^{2t} \\ c_2 e^{2t} \end{bmatrix}, \quad c_1, c_2 \in \mathbb{R}.$$

d) $$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} f_1(0) \\ f_2(0) \end{bmatrix} = \begin{bmatrix} c_1 e^0 + c_2 e^{2\cdot 0} \\ c_2 e^{2\cdot 0} \end{bmatrix} = \begin{bmatrix} c_1 + c_2 \\ c_2 \end{bmatrix}$$

$\Rightarrow c_2 = 1$ and $c_1 = -1$

$$\begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} = \begin{bmatrix} -e^t + e^{2t} \\ e^{2t} \end{bmatrix}$$

Q6

a) β is a basis of V, and consist of 4 vectors, so $\dim(V) = 4$.

b) Coordinates of $\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}$, with respect to basis β:

$$\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} = c_1 \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + c_4 \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$\Updownarrow$

$$c_2 + c_3 + c_4 = 3$$
$$c_1 + c_2 + c_3 = 3$$
$$c_1 + c_3 + c_4 = 3$$
$$c_1 + c_2 + c_4 = 3$$

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 3 \\ 1 & 1 & 1 & 0 & 3 \\ 1 & 0 & 1 & 1 & 3 \\ 1 & 1 & 0 & 1 & 3 \end{bmatrix} \xrightarrow{\text{RREF}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

$$\left[L\left(\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \right) \right]_\beta = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$$

$$L\left(\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \right) = \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix}$$

c) The rank of L is full, so the dimension of the kernel is zero. So $\ker(L) = 0 \Leftrightarrow L$ is injective.