Homework Assignment 3

Complete the following problems without electronic aids.

a) We are given the following real system of linear equations in the unknowns x_1, x_2, x_3 and x_4:

$$\begin{cases} x_1 + 2x_3 + x_4 &= 1 \\ 2x_1 + x_3 &= 2 \\ 3x_1 - x_4 &= 3 \end{cases}$$

Determine the general solution to the system.

b) We are informed that an inhomogeneous system of linear equations over $\mathbb{R}$ with four equations and two unknowns has the two particular solutions

$$\mathbf{v}_p = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \in \mathbb{R}^2 \quad \text{and} \quad \tilde{\mathbf{v}}_p = \begin{bmatrix} -3 \\ 7 \end{bmatrix} \in \mathbb{R}^2.$$

Is the vector $\mathbf{v} = \begin{bmatrix} 4 \\ -5 \end{bmatrix} \in \mathbb{R}^2$ a solution to the corresponding homogeneous system of linear equations?

c) Let W be spanned by the following vectors in $\mathbb{R}^3$:

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 5 \\ 4 \\ 3 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_3 = \begin{bmatrix} 7 \\ 11 \\ 6 \end{bmatrix}.$$

Provide an ordered basis for W.

d) Let V be the real vector space $\mathbb{R}^{3\times 3}$. Provide a subspace of V with dimension 5. Justify your answer.

e) We are given the ordered bases

$$\beta = \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right) \quad \text{and} \quad \gamma = \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \quad \text{for } \mathbb{C}^2.$$

Calculate the change-of-basis matrices ${}_\beta M_\gamma$ and ${}_\gamma M_\beta$. These matrices can also be denoted by ${}_\beta[\mathrm{id}_{\mathbb{C}^2}]_\gamma$ and ${}_\gamma[\mathrm{id}_{\mathbb{C}^2}]_\beta$.

f) The following ordered basis is chosen for the real vector space $\mathbb{R}^{2\times 2}$:

$$\beta = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right).$$

We are given a linear map $M : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ defined by

$$M(\mathbf{A}) = \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix} \cdot \mathbf{A}, \quad \mathbf{A} \in \mathbb{R}^{2\times 2}.$$

Calculate the mapping matrix ${}_\beta[M]_\beta$.

g) Let $C_\infty(\mathbb{R})$ be the real vector space from Example 10.4.5 in the textbook. We define a function $L : C_\infty(\mathbb{R}) \to C_\infty(\mathbb{R})$ by $L(f) = f' + f - 1$, where the expression f' denotes the derivative function of f. Is L a linear map?

h) We are given the following matrix

$$\begin{bmatrix} 2 & 0 & 0 \\ 2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix} \in \mathbb{R}^{3\times 3}.$$

Determine the eigenvalues of the matrix, and provide ordered bases for the corresponding eigenspaces.

The homework assignment must be uploaded to the DTU Learn module for the course via "Assignments". Deadline is Sunday November 22 at 23:55.