

Solution, ordinary exam

①

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Shsp

1. (a)

Q	P	$\neg Q$	$Q \Leftrightarrow P$	$\neg(Q \Leftrightarrow P)$	$\neg(Q \Leftrightarrow P) \wedge \neg Q$	$P \wedge \neg Q$
T	T	F	T	F	F	F
T	F	F	F	T	F	F
F	T	T	F	T	T	T
F	F	T	T	F	F	F

(b)

Same truth tables
so $\neg(Q \Leftrightarrow P) \wedge \neg Q$ and
 $P \wedge \neg Q$ are logically
equivalent.

2. Given: $p(z) = 2z^3 - 2z^2 - 8z - 12 \in \mathbb{C}[z]$ with root $z_0 = 3$.

(a) Polynomial division with $z-3$:

$$\begin{array}{r} z-3 \mid 2z^3 - 2z^2 - 8z - 12 \quad | \underline{2z^2 + 4z + 4} \\ \underline{2z^3 - 6z^2} \\ 4z^2 - 8z - 12 \\ \underline{4z^2 - 12z} \\ 4z - 12 \\ \underline{4z - 12} \\ 0 \end{array}$$

So, $p(z) = (z-3)(2z^2 + 4z + 4)$, where $\deg(z-3) = 1$ and $\deg(2z^2 + 4z + 4) = 2$.

(b) Roots in $p(z)$ are solutions to $p(z) = 0$, so using the rule of zero product, we solve the quadratic equation:

$$\begin{aligned} 2z^2 + 4z + 4 &= 0 \\ \Downarrow \\ z &= \frac{-4 \pm \sqrt{4^2 - 4 \cdot 2 \cdot 4}}{2 \cdot 2} = -1 \pm \frac{\sqrt{16 - 32}}{4} = -1 \pm \frac{\sqrt{-16}}{4} = -1 \pm i \end{aligned}$$

All roots of $p(z)$ are $\underline{-1+i}$, $\underline{-1-i}$, $\underline{3}$

3.

$$\underline{A} = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} \in \mathbb{C}^{2 \times 2} \quad \text{where } a \in \mathbb{C} \text{ is given}$$

(a)

$$\underline{A}^2 = \underline{A} \underline{A} = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} = \begin{bmatrix} a^2 & 2a \\ 0 & a^2 \end{bmatrix}$$

$$\underline{A}^3 = \underline{A}^2 \underline{A} = \begin{bmatrix} a^2 & 2a \\ 0 & a^2 \end{bmatrix} \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} = \begin{bmatrix} a^3 & 3a^2 \\ 0 & a^3 \end{bmatrix}$$

(b)

Claim: $\underline{A}^n = \begin{bmatrix} a^n & n a^{n-1} \\ 0 & a^n \end{bmatrix}$ for all $n \in \mathbb{Z}_{\geq 2}$

Showing via induction on n .

Base case, $n=2$:

$$\underline{A}^2 = \begin{bmatrix} a^2 & 2a^{2-1} \\ 0 & a^2 \end{bmatrix} = \begin{bmatrix} a^2 & 2a \\ 0 & a^2 \end{bmatrix} \text{ fits, so base case is fulfilled.}$$

Induction step, $n > 2$:

Induction hypothesis: $\underline{A}^{n-1} = \begin{bmatrix} a^{n-1} & (n-1)a^{n-1-1} \\ 0 & a^{n-1} \end{bmatrix} = \begin{bmatrix} a^{n-1} & (n-1)a^{n-2} \\ 0 & a^{n-1} \end{bmatrix}$

$$\underline{A}^n = \underline{A}^{n-1} \underline{A} = \begin{bmatrix} a^{n-1} & (n-1)a^{n-2} \\ 0 & a^{n-1} \end{bmatrix} \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} = \begin{bmatrix} a^{n-1}a & a^{n-1} + (n-1)a^{n-2}a \\ 0 & a^{n-1}a \end{bmatrix}$$

$$= \begin{bmatrix} a^{n-1+1} & a^{n-1} + (n-1)a^{n-2+1} \\ 0 & a^{n-1+1} \end{bmatrix} = \begin{bmatrix} a^n & a^{n-1} + (n-1)a^{n-1} \\ 0 & a^n \end{bmatrix}$$

$$= \begin{bmatrix} a^n & n a^{n-1} \\ 0 & a^n \end{bmatrix}, \text{ so fulfilled.}$$

As per the induction principle, the claim is hence fulfilled for all $n \in \mathbb{Z}_{n \geq 2}$.

4.

$$\underline{B} = \begin{bmatrix} 1 & 3 & -3 \\ 2 & 1 & 4 \\ 2 & -1 & 8 \end{bmatrix} \in \mathbb{R}^{3 \times 3}$$

(a)

Kernel: $[\underline{B} \mid \underline{0}] = \left[\begin{array}{ccc|c} 1 & 3 & -3 & 0 \\ 2 & 1 & 4 & 0 \\ 2 & -1 & 8 & 0 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 3 & -3 & 0 \\ 0 & -5 & 10 & 0 \\ 0 & -7 & 14 & 0 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$

so $\ker \underline{B} = \left\{ \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} t \mid t \in \mathbb{R} \right\}$ and a basis is $\left\{ \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} \right\}$.

(b)

From the rref(\underline{B}) above we see that the first and second columns are linearly independent, so a basis for the column space is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \right\}$.

(c)

For a linear map $L: \mathbb{R}^3 \rightarrow \mathbb{R}^3$, $L(\underline{v}) = \underline{B}\underline{v}$, the column space of \underline{B} equals the image space of L , $\text{colsp } \underline{B} = \text{image } L$. We have $\dim_{\mathbb{R}}(\text{image } L) = \dim(\text{colsp } \underline{B}) = 2 \neq \dim_{\mathbb{R}}(\mathbb{R}^3) = 3$, so $\text{image } L$ is not equal to the co-domain, and L is not surjective.

5. Given ordered basis for $V = \mathbb{R}^3$:

(3)

$$\gamma = \left(\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \right)$$

and β is the ordered standard basis

(a) ${}_{\beta} [\text{id}_{\mathbb{R}^3}]_{\gamma} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

← γ -basis vectors in β -coordinates as columns

(b) ${}_{\gamma} [\text{id}_{\mathbb{R}^3}]_{\beta} = \left({}_{\beta} [\text{id}_{\mathbb{R}^3}]_{\gamma} \right)^{-1}$
 $= \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$

The inverse:

$${}_{\beta} [\text{id}]_{\gamma} | \mathbb{I}_3 = \left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array} \right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \end{array} \right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \end{array} \right]$$

$$\underbrace{\hspace{10em}}_{({}_{\beta} [\text{id}]_{\gamma})^{-1}}$$

6. $\begin{cases} f_1'(t) = f_1(t) + 2f_2(t) \\ f_2'(t) = 2f_1(t) + f_2(t) \end{cases} \Leftrightarrow \begin{bmatrix} f_1'(t) \\ f_2'(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}}_{\underline{A}} \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}$

(a) No forcing functions, so the system is homogeneous.

(b)

Eigenvalues are found by solving the characteristic eq:

$$\det(\underline{A} - \lambda \underline{I}_2) = \det \left(\begin{bmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{bmatrix} \right) = (1-\lambda)^2 - 4 = \lambda^2 - 2\lambda - 3 = 0$$

$$\Leftrightarrow \lambda = \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \cdot 1 \cdot (-3)}}{2 \cdot 1} = 1 \pm \frac{\sqrt{4+12}}{2} = 1 \pm \frac{4}{2}, \text{ so } \lambda_1 = 3, \lambda_2 = -1.$$

Eigenvector to $\lambda_1 = 3$:

$$(\underline{A} - 3 \cdot \underline{I}_2) \underline{v}_1 = \underline{0} \Leftrightarrow \left[\begin{array}{cc|c} -2 & 2 & 0 \\ 2 & -2 & 0 \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array} \right], \text{ so an eigenvector is } \underline{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Eigenvector to $\lambda_2 = -1$:

$$(\underline{A} - (-1) \underline{I}_2) \underline{v}_2 = \underline{0} \Leftrightarrow \left[\begin{array}{cc|c} 2 & 2 & 0 \\ 2 & 2 & 0 \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right], \text{ so an eigenvector is } \underline{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

With two linearly independent eigenvectors v_1, v_2 , (since they belong to different eigenspaces), then A is diagonalizable, and the general, real-valued solution to the diff. eq. system is:

$$\begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} = c_1 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad c_1, c_2 \in \mathbb{R}$$

Ⓒ Given initial-value condition: $f_1(0) = 3$ and $f_2(0) = 5$.

$$\begin{bmatrix} f_1(0) \\ f_2(0) \end{bmatrix} = c_1 e^{3 \cdot 0} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-0} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$\begin{cases} c_1 - c_2 = 3 \\ c_1 + c_2 = 5 \end{cases} \Leftrightarrow \begin{bmatrix} 1 & -1 & | & 3 \\ 1 & 1 & | & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & | & 3 \\ 0 & 2 & | & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & | & 4 \\ 0 & 1 & | & 1 \end{bmatrix}$$

$$\Leftrightarrow \begin{cases} c_1 = 4 \\ c_2 = 1 \end{cases}$$

So, the conditioned solution is:

$$\begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} = 4e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$
