Technical University of Denmark



Multiple-choice test exam, November 2023

Course name: Mathematics 1a (Polytechnical foundation) Course no. 01001 and 01003
Exam duration: 2 hours

Aid: All by DTU permitted aid.

"Weighting": All questions in this test exam are weighted equally. This part of the test exam constitutes 50% of the entire test exam.

**Additional information**: The questions are posed first in English, after that in Danish. All questions are multiple choice and no explanatory text or calculations will be taken into account. *Note: This test exam is a transcipt; the test exam will be digital and is to be answered on the DTU Digital Exam platform.*

Solutions, Nov. 2023
Shsp

---

**Question 1**

Given is the following system of linear equations over $\mathbb{R}$ in the three unknowns $x_1, x_2$, and $x_3$:

$$\begin{cases} x_2 + x_3 &= 1 \\ x_1 - 2x_2 + x_3 &= 4 \end{cases}.$$

Which of the below sets denotes the complete solution to the system?

$\cancel{(1)}$ $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}, t \in \mathbb{R}$$

(2) $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ 0 \end{bmatrix}$$

(3) $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, t \in \mathbb{R}$$

(4) $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}, t \in \mathbb{R}$$

(5) $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, t \in \mathbb{R}$$

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & 1 \\ 1 & -2 & 1 & 4 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 3 & 6 \\ 0 & 1 & 1 & 1 \end{array}\right]$$

$$\Leftrightarrow \begin{cases} x_1 = 6 - 3t \\ x_2 = 1 - t \\ x_3 = t \end{cases} \quad \Leftrightarrow \quad \underline{x} = \begin{bmatrix} 6 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix}, t \in \mathbb{R}$$

## Question 2

We are given:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}, \text{ and } \mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Which of the following expressions is identical to the expression $(\mathbf{A} \cdot \mathbf{B})^T \cdot \mathbf{v}$?

(1) $\begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}$

(2) $\begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}$

(3) $\begin{bmatrix} 5 & 11 \\ 2 & 4 \\ -1 & -3 \end{bmatrix}$

(4) $\begin{bmatrix} 5 \\ 11 \end{bmatrix}$

(5) $\begin{bmatrix} 1 \\ 6 \end{bmatrix}$

$$(\underline{\underline{A}}\ \underline{\underline{B}})^T \cdot \underline{v}$$

$$= \underline{\underline{B}}^T \underline{\underline{A}}^T \underline{v}$$

$$= \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}$$

**Question 3**

We are given

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 9 \\ 1 & a & 1 \\ 0 & -1 & a \end{bmatrix} \in \mathbb{R}^{3\times 3}.$$

For which values of $a \in \mathbb{R}$ is matrix $\mathbf{A}$ invertible?

~~(1)~~ $a \in \mathbb{R} \setminus \{-2, 4\}$

(2) $a \in \{-2, 4\}$

(3) $a \in \mathbb{R}$

(4) $a \in \mathbb{R} \setminus \{0\}$

(5) $a \in \mathbb{R} \setminus \{0, 1\}$

$$\det(\underline{\underline{A}}) = a^2 - 2a - 8 = 0$$

$$\Updownarrow \quad a = \begin{cases} -2 \\ 4 \end{cases}$$

Invertible for $\det(\underline{\underline{A}}) \neq 0$, so for

$a \in \mathbb{R} \setminus \{-2, 4\}$

---

**Question 4**

We are informed that $Q(z)$ is a polynomial of degree 2 with the roots 1 and 3. We are given

$$P(z) = Q(z) \cdot (z^3 - 5z^2 - z + 5),$$

and we are informed that $P(z)$ has the root $z = -1$. What are all roots and corresponding multiplicities of $P$?

(1) $z_1 = -1$ with multiplicity 1, $z_2 = 3$ with multiplicity 1, $z_3 = 5$ with multiplicity 1, and $z_4 = 1$ with multiplicity 2.

(2) $z_1 = -1$ with multiplicity 1, $z_2 = 1$ with multiplicity 1, and $z_3 = 3$ with multiplicity 1.

(3) $z_1 = -1$ with multiplicity 2, $z_2 = 1$ with multiplicity 2, and $z_3 = 3$ with multiplicity 1.

(4) $z_1 = -1$ with multiplicity 1, $z_2 = 3$ with multiplicity 1, $z_3 = 5$ with multiplicity 1, and $z_4 = 1$ with multiplicity 1.

(5) $z_1 = 0$ with multiplicity 1, $z_2 = 1$ with multiplicity 1, $z_3 = 3$ with multiplicity 1, and $z_4 = 5$ with multiplicity 1.

$$P(z) = Q(z)(z^3 - 5z^2 - z + 5)$$
$$= q_2 (z-1)(z-3)(z+1) R(z)$$

, where $q_2$ is the highest-degree coefficient in $Q(z)$

Finding $R(z)$ via division by $(z+1)$:

$R(z) = z^2 - 6z + 5$ , with roots $z = 1, 5$

All roots: $-1, 1, 3, 5$ where $am(1) = 2$ and the rest have an am of 1

**Question 5**

Let

$$\mathbf{B} = \begin{bmatrix} 4 & -1 & -1 \\ 6 & -3 & -1 \\ -6 & 5 & 3 \end{bmatrix} \in \mathbb{R}^{3\times 3}.$$

Choose matrices $\Lambda$ and $\mathbf{V}$ such that $\mathbf{V}^{-1}\mathbf{B}\,\mathbf{V} = \Lambda$.

~~(1)~~ $\mathbf{V} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix}, \Lambda = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

(2) $\mathbf{V} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix}, \Lambda = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

(3) $\mathbf{V} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix}, \Lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(4) $\mathbf{V} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix}, \Lambda = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -4 \end{bmatrix}$

(5) No matrices $\mathbf{B}$ and $\Lambda$ that fulfill the above relation exist.

Eigenvalues and corresponding eigenvectors:

$\lambda_1 = 4$ $\quad\quad$ $\lambda_2 = -2$ $\quad\quad$ $\lambda_3 = 2$

$\underline{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ $\quad\quad$ $\underline{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ $\quad\quad$ $\underline{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

The order can be chosen freely but must match between columns of $\underline{V}$ and diag-elements of $\underline{\underline{\Lambda}}$.

**Question 6**

Let $V$ be a subspace of $\mathbb{R}[Z]$, and let $V$ be equipped with the ordered basis $\alpha = (1, Z, Z^2)$. We are given the following information about a linear map $f : V \to V$:

$$f(1) = 1 + Z,\ f(Z) = 1 - Z, \text{ and } f(Z^2) = Z + Z^2.$$

What are the eigenvalues of this map?

~~(1)~~ $1, \sqrt{2}, -\sqrt{2}$

(2) $1, -1, 1$

(3) $-1, -\sqrt{2}, \sqrt{2}$

(4) $\sqrt{2}, -1, 1$

(5) $-\sqrt{2}, 0, 1$

$$[f(1)]_\alpha = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},\quad [f(Z)]_\alpha = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix},\quad [f(Z^2)]_\alpha = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$${}_\alpha[f]_\alpha = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \leftarrow \text{Mapping matrix}$$

Eigenvalues: $\lambda = \begin{cases} -\sqrt{2} \\ \sqrt{2} \\ 1 \end{cases}$

**Question 7**

A real system of differential equations is given by

$$\begin{bmatrix} f_1'(t) \\ f_2'(t) \end{bmatrix} = \mathbf{A} \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}, \text{ where } \mathbf{A} = \begin{bmatrix} -1 & 3 \\ 3 & -1 \end{bmatrix}.$$

We are informed that $\begin{bmatrix} f_1(0) \\ f_2(0) \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$. What is the solution?

~~(1)~~ $f_1(t) = 3e^{2t} + 3e^{-4t}$, $f_2(t) = 3e^{2t} - 3e^{-4t}$

(2) $f_1(t) = e^{2t} + 5e^{-4t}$, $f_2(t) = e^{2t} - e^{-4t}$

(3) $f_1(t) = 5e^{2t} + e^{-4t}$, $f_2(t) = 5e^{2t} - 5e^{-4t}$

(4) $f_1(t) = 6e^{2t}$, $f_2(t) = 0$

(5) $f_1(t) = e^{2t}$, $f_2(t) = e^{-4t}$

Eigenvalues and -vectors of $\underline{\underline{A}}$:

$\lambda_1 = -4$ $\qquad\qquad$ $\lambda_2 = 2$

$\underline{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $\qquad\qquad$ $\underline{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

General solution

$$\begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} = c_1 e^{-4t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad c_1, c_2 \in \mathbb{R}$$

Conditioned solution

$$c_1 e^{-4\cdot 0} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{2\cdot 0} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix} \Leftrightarrow \begin{cases} c_1 = -3 \\ c_2 = 3 \end{cases}$$

$$\begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} = -3e^{-4t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + 3e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

**Question 8**

A polynomial is given by

$$P(Z) = Z^2 + aZ + b\ ,\ a,b \in \mathbb{C}.$$

We are informed that $z_1 = 1+i$ and $z_2 = 2i$ are roots in $P$. What are the values of $a$ and $b$?

(1) $a = -3i-1\ ,\ b = -2+2i$

(2) $a = 1+i\ ,\ b = 2i$

(3) $a = 3i+1\ ,\ b = 2-2i$

(4) $a = 3i-1\ ,\ b = -2+2i$

(5) $a = -3i-1\ ,\ b = -2-2i$

END OF THE EXAM

$$\begin{cases} (1+i)^2 + a(1+i) + b = 0 \\ (2i)^2 + a(2i) + b = 0 \end{cases}$$

$$\Updownarrow$$

$$\begin{cases} a(1+i) + b = -2i \\ a(2i) + b = 4 \end{cases}$$

$$\Updownarrow$$

$$\left[\begin{array}{cc|c} 1+i & 1 & -2i \\ 2i & 1 & 4 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & -1-3i \\ 0 & 1 & -2+2i \end{array}\right]$$