MC exam Math 1a 01003

The following approach to scoring responses is implemented and is based on "One best answer":

There is always only one correct answer – a response that is more correct than the rest
Students are only able to select one answer per question
Every correct answer corresponds to 1 point
Every incorrect answer corresponds to 0 points (incorrect answers do NOT result in subtraction of points)

Solutions
5/12-23
shsp

Let the matrices $\mathbf{A}$ and $\mathbf{B}$ be given by:

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 3 & 0 \\ 0 & 7 & -2 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

Which of the below numbers state the determinant of $\mathbf{A} \cdot \mathbf{B}$?

Vælg en svarmulighed

☒ $\det(\mathbf{A} \cdot \mathbf{B}) = 42$

○ $\det(\mathbf{A} \cdot \mathbf{B}) = -42$

○ $\det(\mathbf{A} \cdot \mathbf{B}) = -21$

○ $\det(\mathbf{A} \cdot \mathbf{B}) = 0$

○ $\det(\mathbf{A} \cdot \mathbf{B}) = 21$

$$\det(\underline{\underline{A}} \cdot \underline{\underline{B}}) = \det\left(\begin{bmatrix} 2 & -2 & 9 \\ 1 & 6 & 0 \\ 0 & 14 & -6 \end{bmatrix}\right)$$

$$= \sum_{i=1}^{3} (-1)^{i+3} a_{i3} \det(\underline{\underline{A}}(i;3)) \xleftarrow{} \text{Expanding by column } j=3$$

$$= (-1)^4 \cdot 9 \cdot \det\left(\begin{bmatrix} 1 & 6 \\ 0 & 14 \end{bmatrix}\right) + 0 + (-1)^6 \cdot (-6) \cdot \det\left(\begin{bmatrix} 2 & -2 \\ 1 & 6 \end{bmatrix}\right)$$

$$= 9 \cdot 14 - 6 \cdot 14$$

$$= 3 \cdot 14$$

$$= 42$$

Let $\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$ be an invertible matrix, and let $\mathbf{b} = \begin{bmatrix} 4 \\ 12 \end{bmatrix}$ be a column vector.
Which column vector $\mathbf{x} \in \mathbb{R}^2$ is a solution to the equation $\mathbf{Ax} = \mathbf{b}$?

Vælg en svarmulighed

☒ $\mathbf{x} = \begin{bmatrix} 10 \\ 6 \end{bmatrix}$

○ $\mathbf{x} = \begin{bmatrix} -10 \\ 6 \end{bmatrix}$

○ $\mathbf{x} = \begin{bmatrix} 10 \\ -6 \end{bmatrix}$

○ $\mathbf{x} = \begin{bmatrix} 6 \\ 10 \end{bmatrix}$

○ $\mathbf{x} = \begin{bmatrix} -6 \\ -10 \end{bmatrix}$

$$[\underline{\underline{A}} \mid \underline{b}] = \left[\begin{array}{cc|c} 1 & -1 & 4 \\ 0 & 2 & 12 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 10 \\ 0 & 1 & 6 \end{array}\right]$$

so $\underline{x} = \begin{bmatrix} 10 \\ 6 \end{bmatrix}$

We are given the second-degree polynomial:

$$p(z) = 3z^2 - 6z + 6 \ , \ z \in \mathbb{C}\,.$$

Which of the below options state the discriminant d of the polynomial as well as one of the roots r of the polynomial?

Vælg en svarmulighed

○ $d = -36, r = i - 1$

☒ $d = -36, r = 1 - i$

○ $d = -6i, r = 1 - i$

○ $d = 6i, r = 1 + i$

○ $d = 0, r = 1 + i$

$$d = (-6)^2 - 4 \cdot 3 \cdot 6 = 36 - 72 = -36$$

$$p(z) = 0$$

$$\Updownarrow$$

$$z = \frac{6 \pm \sqrt{-36}}{6} = \frac{6 \pm i6}{6} = 1 \pm i$$

Consider the real differential equation:

$f'(t) = \frac{1}{t} f(t) + t$ where $t > 0$.

Which of the below expressions is a particular solution to the differential equation?

Vælg en svarmulighed

☒ $f(t) = t^2 \ , \ t > 0$

○ $f(t) = t^2 + ce^t \ , \ t > 0 \, , c \in \mathbb{R}$

○ $f(t) = t(t^2 + t) \ , \ t > 0$

○ $f(t) = t^2 - 1 \ , \ t > 0$

○ $f(t) = ce^t + 1 \ , \ t > 0 \, , c \in \mathbb{R}$

$$
\begin{aligned}
f(t) &= e^{\ln(t)} \int e^{-\ln(t)} \cdot t \, dt \\
&= t \int \frac{1}{t} \cdot t \, dt \\
&= t \int 1 \, dt \\
&= t(t + c) \\
&= t^2 + t \cdot c \qquad \text{where } c \in \mathbb{R}
\end{aligned}
$$

For $c = 0$ we have the answer.

Consider the real second-order differential equation:

$$f''(t) - 6f'(t) + 9f(t) = 0$$

Which of the below expressions is ***NOT*** a solution to the differential equation?

Vælg en svarmulighed

- ○ $f(t) = te^{3t}\ ,\ t \in \mathbb{R}$
- ☒ $f(t) = \cos(3t) + \sin(3t)\ ,\ t \in \mathbb{R}$
- ○ $f(t) = 4e^{3t}\ ,\ t \in \mathbb{R}$
- ○ $f(t) = e^{3t}(t-1)\ ,\ t \in \mathbb{R}$
- ○ $f(t) = 0\ ,\ t \in \mathbb{R}$

Characteristic equation:

$$\lambda^2 - 6\lambda + 9 = 0 \quad , \quad d = (-6)^2 - 4 \cdot 1 \cdot 9$$

$$= 36 - 36$$

$$= 0$$

so, double root.

$$\Updownarrow$$

$$\lambda = \frac{-(-6) \pm \sqrt{d}}{2 \cdot 1} \leftarrow$$

$$= \frac{6}{2} = 3 \quad \text{with} \quad am(3) = 2.$$

General solution:

$$f(t) = c_1 e^{3t} + c_2 e^{3t} t \quad , \quad c_1 , c_2 \in \mathbb{R}$$

Let a matrix $\mathbf{A}$ be given by:

$$\mathbf{A} = \begin{bmatrix} 2 & 3 & 5 \\ 0 & 0 & -2 \\ 0 & -2 & 0 \end{bmatrix}.$$

In which of the below expressions are $\mathbf{v}_1$ and $\mathbf{v}_2$ eigenvectors of matrix $\mathbf{A}$?

Vælg en svarmulighed

- ○ $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -4 \\ 2 \\ 2 \end{bmatrix}$
- ○ $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$
- ☒ $\mathbf{v}_1 = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$
- ○ $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$
- ○ $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$

Eigenvalues:

$$\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}_3) = \det\left(\begin{bmatrix} 2-\lambda & 3 & 5 \\ 0 & -\lambda & -2 \\ 0 & -2 & -\lambda \end{bmatrix}\right)$$

$$= \sum_{i=1}^{3} (-1)^{1+i} a_{i1} \cdot \det(\underline{\underline{A}}(i;1))$$

Expanding by column j=1

$$= (2-\lambda)(\lambda^2 - 4) = 0$$

$$\Updownarrow$$

$$\lambda = \begin{cases} 2 \\ -2 \end{cases} \quad \text{where} \quad \begin{matrix} am(2) = 2 \\ am(-2) = 1 \end{matrix}$$

Eigenvectors:

For $\lambda = 2$: $\left[\begin{array}{ccc|c} 0 & 3 & 5 & 0 \\ 0 & -2 & -2 & 0 \\ 0 & -2 & -2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Leftrightarrow \underline{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t, \ t \in \mathbb{R}$

For $\lambda = -2$: $\left[\begin{array}{ccc|c} 4 & 3 & 5 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & -2 & 2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Leftrightarrow \underline{v} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} t, \ t \in \mathbb{R}$

Let $\mathbf{A} \in \mathbb{R}^{3\times 3}$ be a matrix.

Let $L_{\mathbf{A}} : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear map given by: $L_{\mathbf{A}}(\mathbf{v}) = \mathbf{A}\mathbf{v}$.

We are informed that $L_{\mathbf{A}}$ has the eigenvalues $\lambda_1 = 1\ ,\ \lambda_2 = -1\ ,\ \lambda_3 = 2$ with corresponding eigenspaces:

$$E_{\lambda_1} = \operatorname{span}_{\mathbb{R}}\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)\ ,\ E_{\lambda_2} = \operatorname{span}_{\mathbb{R}}\left(\begin{bmatrix}1\\0\\1\end{bmatrix}\right)\ ,\ E_{\lambda_3} = \operatorname{span}_{\mathbb{R}}\left(\begin{bmatrix}-1\\0\\1\end{bmatrix}\right).$$

Choose which of the following matrices that is matrix $\mathbf{A}$.

Vælg en svarmulighed

☒ $\mathbf{A} = \begin{bmatrix}\frac{1}{2} & 0 & -\frac{3}{2}\\ 0 & 1 & 0\\ -\frac{3}{2} & 0 & \frac{1}{2}\end{bmatrix}$

○ $\mathbf{A} = \begin{bmatrix}0 & \frac{1}{2} & -\frac{3}{2}\\ 1 & 0 & 0\\ 0 & -\frac{3}{2} & \frac{1}{2}\end{bmatrix}$

○ $\mathbf{A} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 1 & 1\end{bmatrix}$

○ $\mathbf{A} = \begin{bmatrix}0 & -\frac{3}{2} & \frac{1}{2}\\ 1 & 0 & 0\\ 0 & \frac{1}{2} & -\frac{3}{2}\end{bmatrix}$

○ $\mathbf{A} = \begin{bmatrix}\frac{1}{2} & -\frac{3}{2} & 0\\ 0 & 0 & 1\\ -\frac{3}{2} & \frac{1}{2} & 0\end{bmatrix}$

Inverse $\underline{\underline{V}}^{-1}$;

$$[\underline{\underline{V}} \mid \underline{\underline{I}}_3] = \left[\begin{array}{ccc|ccc}0 & 1 & -1 & 1 & 0 & 0\\ 1 & 0 & 0 & 0 & 1 & 0\\ 0 & 1 & 1 & 0 & 0 & 1\end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc}1 & 0 & 0 & 0 & 1 & 0\\ 0 & 1 & 0 & \frac{1}{2} & 0 & \frac{1}{2}\\ 0 & 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2}\end{array}\right]$$

$$\underbrace{\phantom{\begin{matrix}-\frac{1}{2} & 0 & \frac{1}{2}\end{matrix}}}_{\underline{\underline{V}}^{-1}}$$

$$\underline{\underline{V}}^{-1}\underline{\underline{A}}\,\underline{\underline{V}} = \underline{\underline{\Lambda}}$$

$$\Updownarrow$$

$$\underline{\underline{A}} = \underline{\underline{V}}\,\underline{\underline{\Lambda}}\,\underline{\underline{V}}^{-1}$$

$$= \begin{bmatrix}0 & 1 & -1\\ 1 & 0 & 0\\ 0 & 1 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 2\end{bmatrix}\begin{bmatrix}0 & 1 & 0\\ \frac{1}{2} & 0 & \frac{1}{2}\\ -\frac{1}{2} & 0 & \frac{1}{2}\end{bmatrix}$$

$$= \begin{bmatrix}0 & -1 & -2\\ 1 & 0 & 0\\ 0 & -1 & 2\end{bmatrix}\begin{bmatrix}0 & 1 & 0\\ \frac{1}{2} & 0 & \frac{1}{2}\\ -\frac{1}{2} & 0 & \frac{1}{2}\end{bmatrix}$$

$$= \begin{bmatrix}\frac{1}{2} & 0 & -\frac{3}{2}\\ 0 & 1 & 0\\ -\frac{3}{2} & 0 & \frac{1}{2}\end{bmatrix}$$